BEAMS ON NON LINEAR FOUNDATIONS

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THESIS

BEAMS ON NONLINEAR FOUNDATIONS

by

Vo Thanh Tam

September 1975

Thesis Advisor: D. Salinas

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Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Beams on Nonlinear Foundations		5. TYPE OF REPORT & PERIOD COVERED Master's Thesis September 1975
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Vo Thanh Tam		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		12. REPORT DATE September 1975
		13. NUMBER OF PAGES
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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(Page 1) S/N 0102-014-6601 |

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

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Beams on Nonlinear Foundations

by

Vo Thanh Tam

Submitted in partial fulfillment of the
requirements for the degree of

MECHANICAL ENGINEER

and

MASTER OF SCIENCE IN MECHANICAL ENGINEERING

from the

NAVAL POSTGRADUATE SCHOOL
September 1975

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The solution of a beam on an elastic nonlinear foundation by the Finite Element Method is presented in this thesis. Discontinuous (Winkler) and continuous foundations are considered. The general formulation is based on the Galerkin method. The solution technique for the linear case uses Gauss elimination and the solution technique for the nonlinear case is based on Brown's method (a modified Newton-Raphson). Some illustrative examples are presented. A brief comparison of continuous and non-continuous foundations is made.

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LIST OF SYMBOLS AND NOTATIONS

b_1, b_2, b_3	Coefficients of a least square fit polynomial
e	Superscript referred to an "element"
EI	Beam flexural rigidity
F.D.M.	Finite Difference Method
F.E.M.	Finite Element Method
G_i	A set of shape functions
ITMAX	Maximum number of iteration for nonlinear problems
k	Foundation modulus
l	Beam element length
L	Beam total length
q	Applied load
Q_i	Force vector
r	Reaction of the foundation
s	Shear force in the foundation material
V	Displacement
V_i	Displacement Vector
C_{ik}, K_{ik}, M_{ik}	Coefficient matrices of V_i
N_{ijk}	Coefficient matrix of $V_i V_j$
$(\ \ )^n$	nth order differentiation of ( ) w.r.t.x., where n is the number of primes
$\sum$	Summation

$\langle a, b \rangle$	Integral of the function $a(x) \cdot b(x)$
[ ]	Square matrix

ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Dr. David Salinas, Associate Professor of Mechanical Engineering, for his advice and guidance throughout the course of development of this text. Without his generous cooperation this work could not have been accomplished.

The author is obligated to Dr. Robert S. Andrews, Assistant Professor of Oceanography, for his kindness in assisting the writer to understand the nature and properties of soil structures, without which this work would not have any practical meaning.

I. INTRODUCTION

In treating the various problems related to structures on elastic foundations, it has been customary for engineers to assume that the pressure in the foundation is linearly proportional at every point to the deflection of the foundation at that point, independent of the pressure or deflection occurring in other parts of the foundation. This assumption, is mathematically by far the simplest that one can make regarding the nature of a supporting elastic medium. It assumes a complete lack of continuity in the material of the foundation as if it consisted of a series of independent springs which deflect when directly loaded. This theory has proved adequate for calculating stresses and deflections in railroad tracks and has found many applications to plate and shell structures [Ref. 1]. However no particular claim has been made that the deformation or pressure distribution in actual earth foundations could be predicted by this method.

In recent years, some authors have treated foundations as a continuous medium which, in contrast to the Winkler hypothesis, represents the case of complete continuity in the material [Ref. 9]. It is evident that the two types of foundation models lead to different results, but how quantitatively significant is the difference has not previously been made clear.

The properties of actual foundations, however, seem to lie somewhere in the gap between these two extreme cases. The physical properties of soils are obviously of a very complicated nature. Their deformation behavior is influenced by a number of factors such as the physical structure, porosity, existence and movement of fluids in the pores, etc. In addition, such geologic features as faults, joints, seams, crushed zones, fissures and other tectomic effects produce behavior significantly different from that derived on the assumption of continuous mass [Ref. 14]. The question is then raised: does the approximation based on the Winkler hypothesis still hold over various types of soil foundations? If it does not, then how much error may be expected? How much does the shearing effect in the material of the foundation contribute to its deformation? Considering only the deflections of one-dimensional beams on foundations (for simplicity), this thesis attempts to answer these questions.

II. DESCRIPTION OF ELASTIC FOUNDATIONS

This chapter presents brief descriptions of the Winkler, Modified Winkler and Continuous foundations considered in this thesis. Many other foundations have been proposed over the years by numerous investigators. Kerr [Ref. 31] presents a summary of a number of linear foundations.

A. DEFINITION OF WINKLER, MODIFIED WINKLER AND CONTINOUS FOUNDATIONS

The following definitions will be used throughout this thesis.

1. Classical Winkler Foundation

This type of foundation is characterized by the fact that the deflection at every point of the foundation is linearly proportional to the pressure applied at that point and is independent of pressures or deflections acting elsewhere in the foundation. This assumption is equivalent to considering the foundation as a discontinuous medium composed of independent elastic springs. It is believed that this hypothesis was proposed in 1867 by Winkler[1] [Refs. 27, 9, 29].

2. Modified Winkler Foundation

This type of foundation, still a discontinuous medium, is

[1] Some authors claim that Euler seems to have been the first to formulate the hypothesis, although it is generally attributed to Winkler.

however characterized by the fact that the pressure at every point is nonlinearly proportional to the deflection at that point.

In this thesis, reference to a Winkler foundation means either the classical (linear) Winkler foundation or the modified (nonlinear) Winkler foundation.

3. Continuous Foundation

In contrast to the Winkler foundation, the pressure and deflection at a point in a continuous foundation is affected by the behavior of the entire foundation [Refs. 1, 9, 27].

4. Mixed Mode Foundation

The real foundations of natural soils are more complex than either the Winkler or continuous foundations. They are neither completely continuous nor discontinuous [Refs. 14, 6]. The pressure and deflection at any point are nonlinearly dependent. This type of foundation is therefore considered as a combination of the modified Winkler foundation and the continuous foundation.

B. MATHEMATICAL MODELS

In this chapter the governing equations of the foundations considered in this thesis are developed.

Consider an elastic foundation which is a compressible single layer of thickness H placed on a rigid base. It will be assumed that the thickness of the elastic foundation, its support conditions, the elastic constraints and all other properties as well as the external load do not

vary in the z-direction. A narrow plate of thickness w is cut from the elastic foundation by two planes parallel to the x y plane. A beam of the same thickness w rests on the surface of this elastic foundation.

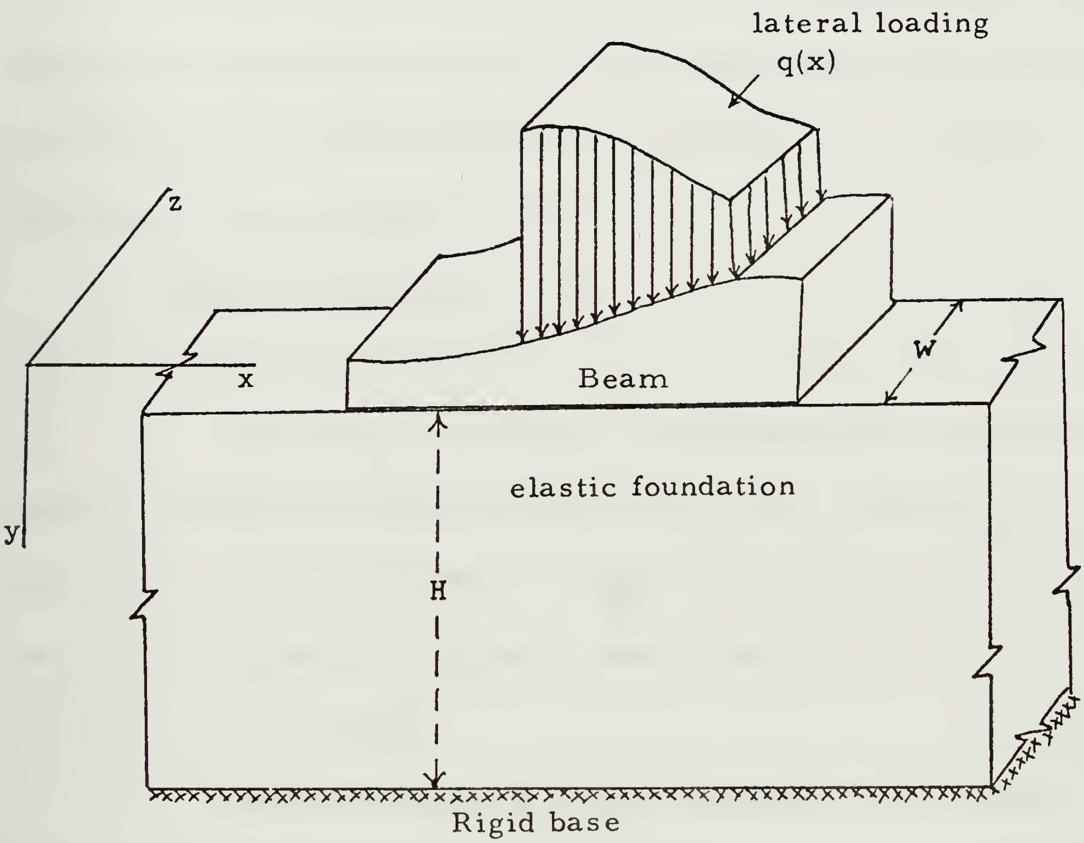


Figure 1. Geometry of an elastically supported beam

Let an external load q(x), pounds per unit length, act on the beam (Figure 1). If r(x) is the reaction of the elastic foundation against the beam, the differential equation of bending for the beam is then [Ref. 8]:

$$(EIV'')'' = q(x) - r(x) \tag{1}$$

where EI (x) and V (x) are the flexural rigidity and the deflection of the beam, respectively.

Equation (1) is true for any type of foundation. It contains two unknown functions, V (x) and r (x). In order to determine them, an additional relationship is needed. This relationship is associated with the response of the elastic foundation and depends on the type of foundation being considered.

1. Winkler Foundations

a. Governing Equation

The Winkler foundation is characterized by a foundation modulus, generally denoted by the letter k^*. Let V, inches and r, pounds per inch, be the deflection and the reaction of the foundation, respectively. The Winkler foundation is characterized by the equation

$$r(x) = k^* V^p(x) \tag{2}$$

For the classical Winkler foundations, $p = 1$ and k^* is a constant and is expressed in pounds per cubic inch. This assumption has proved adequate for calculating stresses and deflection in railroad tracks [Ref. 1].

b. Values of Foundation Modulii

For sand, k^* may vary from 25 to 100. lb/cu.in. For ordinary soils on which railroad tracks have been built, k^* may vary from 110. to 130. lb/cu.in. For gravel, the value of k^* is even more uncertain and it can vary from 200. to 1200. lb/cu.in. [Refs. 1, 27].

2. Modified Winkler Foundations

For actual foundations of natural soils, k^* is the coefficient associated with the nonlinear term [Ref. 6]. Therefore, to generalize the Winkler hypothesis, let p be any positive real number. Such foundations are called modified Winkler foundations.

3. Continuous Foundations

a. Governing Equation

Let s be the shear force in the foundation, pounds per inch, and k^* be the foundation modulus, pounds per cubic inch. The shear force s is a measure of the friction force between soil particles. The governing equation of continuous foundations is [Ref. 9]:

$$r(x) = k^* V(x) - (s\, V'(x))' \qquad (3)$$

Details of the continuous foundation analysis are given in Appendix A.

b. Values of s and k

The values of s and k depend on the contact area of beam and foundation per unit length of beam. For a loaded area of 6.5 inches of width, equal to that of a 12WF27 standard beam, and 1. inch of length, typical values of s and k for some soils are:

Average sea floor sediment:

$s = 18.6$ lb $\qquad k = 8.3\ \text{lb/in}^2$

A typical beach soil:

$s = 1000.$ lb $\qquad k = 500.\ \text{lb/in}^2$

where $k = wk^*$

A typical inland soil:

$s = 6000.\ \mathrm{lb}$ $\qquad k = 3000.\ \mathrm{lb/in}^2$

Additional values of soil characteristics are given in Reference 7.

4. Mixed Mode Foundations

a. Governing Equation

The governing equation of this type of foundations can be derived from the definition of the mixed mode foundations given in part A. The mixed mode foundation is a combination of a modified Winkler foundation and a continuous foundation:

$$r(x) = k\,V^p(x) - (s\,V'(x))' \tag{4}$$

It is seen that any of the previous foundations are special cases of the mixed mode foundation. For the Winkler foundation $s = 0$ and $p = 1$, for the nonlinear foundation $s = 0$ and for the continuous foundation $p = 1$.

III. GENERAL FORMULATION OF THE PROBLEM

A. GOVERNING EQUATIONS

Consider a beam on an elastic foundation. The differential equation of bending for a beam on an elastic foundation is given by [Ref. 8]:

$$(EIV''(x))'' = q(x) - r(x) \tag{1}$$

where q(x) is the external loading, V(x) is the beam deflection and r(x) is the reaction of the foundation. The mixed mode foundation, by definition, can be considered as the most general type of foundation. Its governing equation is given Eqn. 1 with

$$r(x) = kV^p(x) - (sV'(x))' \tag{4}$$

and $p > 0$.

Substitution of (4) into (1) gives:

$$(EIV''(x))'' = q(x) - kV^p(x) + (sV'(x))' \tag{5}$$

or

$$(EIV'')'' - (sV')' + kV^p = q \tag{6}$$

To solve this equation for any positive real p, consider a least square best fit polynomial such that

$$V^p(x) = b_1 + b_2 V(x) + b_3 V^2(x) + \ldots \tag{7}$$

where the b_i coefficients $(i = 1, 2, 3, \ldots)$ are constants.

For simplicity in the development of an actual computer program, equation (7) is taken to be a second degree polynomial. This restriction is not inherent in the Galerkin procedure but rather in the Finite Element formulation of Galerkin adopted in this thesis. Substitution of (7) into (6) leads to the following equation:

$$(EIV'')'' - (sV')' + k(b_1 + b_2V + b_3V^2) = q \qquad (8)$$

or

$$(EIV'')'' - (sV')' + kb_2V + kb_3V^2 = q - kb_1 \qquad (9)$$

Letting

$$a_1 = kb_2 \qquad (10)$$

$$a_2 = kb_3 \qquad (11)$$

$$f = q(x) - kb_1 \qquad (12)$$

equation (9) may be rewritten:

$$(EIV'')'' - (sV')' + a_1V + a_2V^2 = f \qquad (13)$$

1. Beam on a Winkler Foundation

Equation (13) can be applied to the case of a beam on a classical Winkler foundation, by letting $b_1 = b_3 = 0$, $b_2 = 1$ and $S_1 = 0$. Hence the governing equation of bending for a beam on a classical Winkler foundation is:

$$(EIV'')'' + a_1V = q(x) \qquad (14)$$

where a_1 is defined by Eqn. 10.

2. Beam on a Modified Winkler Foundation

When s = 0, Equation (13) becomes the governing equation of bending for a beam on a modified Winkler foundation:

$$(EIV'')'' + a_1 V + a_2 V^2 = f \qquad (15)$$

where terms are defined by Eqns. 10, 11, 12.

3. Beam on a Continuous Foundation

Equation (13) can be applied to the case of bending of a beam on a continuous foundation, by letting $b_2 = 1$, $b_1 = b_3 = 0$:

$$(EIV'')'' - (sV')' + a_1 V = q(x) \qquad (16)$$

4. Beam on a Mixed Mode Foundation

Equation (13) is the governing equation of bending for a beam on a mixed mode foundation:

$$(EIV'')'' - (sV')' + a_1 V + q_2 V^2 = f \qquad (17)$$

B. BOUNDARY CONDITIONS

Reference 8 thoroughly develops the boundary conditions for a beam. Only the boundary conditions on displacement and slope (the so called essential or principal boundary conditions) must be imposed in a variational formulation of the problem.

C. ASSUMPTIONS AND RESTRICTIONS

Some restrictions have already been stated in the introduction section of this thesis; other restrictions will now be discussed.

1. The flexural rigidity EI must be "slowly varying" in accordance with the development of the basic equations of classical beam theory.

2. The shear coefficient s must also be "slowly varying" in accordance with the development in Appendix A.

3. The foundation modulus k(x) and the load variable q(x) need not be slowly varying.

D. SOLUTION TECHNIQUES

The Finite Element Method via the Galerkin approach will be used here. Its formulation will be presented in the next section. There are two types of problems to be considered.

1. Linear problems

Consider the linear system

$$(EIV'')'' - (sV')' + kV = q \tag{18}$$

Suppose the solution for $q = q_1$ is V_1 and the solution for $q = q_2$ is V_2, then for $q = q_1 + \alpha q_2$ we have the solution:

$$V = V_1 + \alpha V_2 \tag{19}$$

The application of the Finite Element Method to any linear differential equation leads to a linear system of algebraic equations which will be solved by Gaussian elimination.

2. Nonlinear problems

In contrast to the linear case, any nonlinear differential equation, like (13) or (15), leads to a nonlinear system of algebraic equations which, in this thesis, will be solved by Brown's iteration

method. Reference 13 thoroughly develops the Brown's iteration method which is a modified Newton-Raphson method.

The results of the Finite Element Method solutions presented here will be checked with the classical solutions if available, or with other methods, such as the Finite Difference Method. Programming details will be presented in Chapter V.

IV. FINITE ELEMENT FORMULATION

A. APPROXIMATION BASED ON THE GALERKIN PROCEDURE

In order to obtain a numerical solution for displacements of beams on elastic foundations, let V be approximated by the m degrees of freedom expression:

$$V \approx \sum_{i=1}^{m} V_i \; G_i\,(x) \qquad (21)$$

where $G_i(x)$ are global (or system) shape functions and the unknowns V_i coefficients are generalized coordinates [Ref. 14], i.e., system degrees of freedom. Since the Galerkin method deals directly with the differential equation [Refs. 21, 15, 22], (21) will be inserted in (13) and the equation residual is formed as follows:

$$\sum_{i=1}^{m} \Big[\langle (EIV_iG_i'')'', G_k(x) \rangle - \langle (s \;\; V_iG_i'(x))', G_k(x) \rangle$$

$$+ \langle a_1 \;\; V_iG_i(x), G_k(x) \rangle + \sum_{j=1}^{m} \langle a_2 \;\; V_i \; G_i \;(x)V_jG_j(x), G_k(x) \rangle \Big]$$

$$- \langle f, G_k(x) \rangle = 0$$

where the notation $\langle a, b \rangle$ means $\int_0^L a(x)b(x)dx$. Then

$$\sum_{i=1}^{m} \Big[V_i \;\; \langle (EIG_i''(x))'', G_k(x) \rangle - V_i \;\; \langle (s \; G_i'(x))', G_k(x) \rangle$$

$$+ a_1V_i \; \langle G_i(x), G_k(x) \rangle + \sum_{j=1}^{m} a_2V_iV_j \; \langle G_i(x)G_j(x), G_k(x) \rangle \Big]$$

$$- \langle f, G_k(x) \rangle = 0$$

Integration of the first two terms by parts yields:

$$\sum_{i=1}^{m} [V_i \langle EIG_i''(x), G_k''(x) \rangle + V_i \langle sG_i'(x), G_k'(x) \rangle$$

$$+ a_1 V_i \langle G_i(x), G_k(x) \rangle + \sum_{j=1}^{m} a_2 V_i V_j \langle G_i(x) G_j(x), G_k(x) \rangle]$$

$$- \langle f, G_k(x) \rangle = 0 \qquad (22)$$

There is a constant of integration due to boundary conditions left by the process of integration by parts in Eqn. 22. However this constant can be omitted in the Galerkin procedure [Ref. 23].

Let

$$\left.\begin{aligned} K_{ik} &= \langle EIG_i''(x), G_k''(x) \rangle \\ C_{ik} &= \langle s\, G_i'(x), G_k'(x) \rangle \\ M_{ik} &= \langle G_i(x), G_k(x) \rangle \\ N_{ijk} &= \langle G_i(x)\, G_j(x), G_k(x) \rangle \\ Q_k &= \langle f, G_k(x) \rangle \end{aligned}\right\} \qquad (23)$$

Then, Eqn. 22 may be rewritten as follows:

$$\sum_{i=1}^{m} [K_{ik} + s\, C_{ik} + a_1 M_{ik}] V_i$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{m} a_2 N_{ijk} V_i V_j - Q_k = 0 \qquad (24)$$

for $k = 1, 2, 3, \ldots, m$

This is a system of nonlinear algebraic equations, where the unknown displacements V_i must also satisfy the imposed boundary conditions.

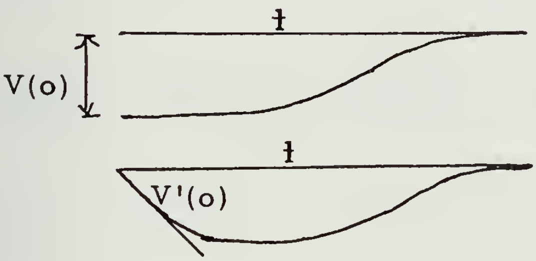
B. ELEMENT SHAPE FUNCTIONS

Consider a beam element. There are two nodal coordinates, the deflection and slope at each end, and therefore each element must have a total of four degrees of freedom. On the element level these coordinates are numbered as follows:

$$\text{Coordinate 1} = V(o) \qquad \text{Coordinate 3} = V(\ell)$$

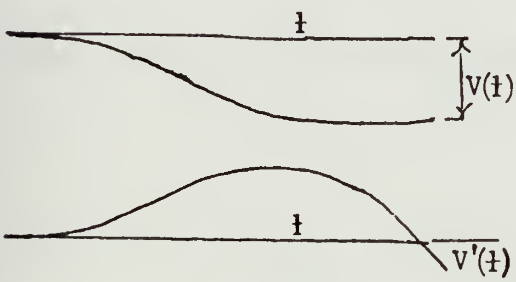
$$\text{Coordinate 2} = V'(o) \qquad \text{Coordinate 4} = V'(\ell)$$

The compatible element shape functions must be cubics and it is evident that four independent shape functions are required [Refs. 14, 16]. Let $G_i^e(x)$, the element shape functions, be defined as follows, where the superscript "e" refers to the beam element:



$$G_1^e = 1 - \frac{3x^2}{\ell^2} + \frac{2x^3}{\ell^3}$$

$$G_2^e = \frac{x^3}{\ell^2} - 2\frac{x^2}{\ell} + x \qquad (25)$$



$$G_3^e = \frac{3x^2}{\ell^2} - 2\frac{x^3}{\ell^3}$$

$$G_4^e = \frac{x^3}{\ell^2} - \frac{x^2}{\ell}$$

Figure 2. Element shape functions

Note that the function $G_i^e(x)$ represents the variation of V along the element in such a way that $G_1^e(o) = G_2^{e'}(o) = G_3^e(\ell) = G_4^{e'}(\ell) = 1$ and that

other evaluations of $G_i^e(o)$, $G_i^e(1)$, $G_i^{e'}(o)$, $G_i^{e'}(1)$ vanish; cf. the sketches in figure 2. where x is the local coordinate from left to right and l denotes the length of the beam element.

C. FORMATION OF ELEMENT MATRICES

Once the element shape functions are defined, the components of the element matrices K_{ik}^e, C_{ik}^e, M_{ik}^e, Q_k^e and N_{ijk}^e can be computed through ordinary integration of polynomials. The following results, associated with the linear terms, have been given in many text books [Refs. 21, 14, 25].

$$K_{ik}^e = \langle EIG_i^{e''}(x),\ G_k^{e''}(x)\rangle = EI^e \begin{bmatrix} \frac{12.}{l^3} & \frac{6.}{l^2} & -\frac{12.}{l^3} & \frac{6.}{l^2} \\ & \frac{4.}{l} & \frac{-6.}{l^2} & \frac{2.}{l} \\ \text{Symmetric} & & \frac{12.}{l^3} & \frac{-6.}{l^2} \\ & & & \frac{4.}{l} \end{bmatrix}$$

where EI^e is a constant (the average EI) over each element.

$$C_{ik}^e = \langle s\, G_i^{e'}(x), G_k^{e'}(x)\rangle = s^e \begin{bmatrix} \frac{1.2}{l} & 0.1 & \frac{-1.2}{l} & 0.1 \\ & .133333l & -0.1 & -0.033333l \\ \text{symmetric} & & \frac{1.2}{l} & -0.1 \\ & & & 0.133333\,l \end{bmatrix}$$

where s^e is a constant (the average) over each element.

$$M^e_{ik} = \left\langle G^e_i(x), G^e_k(x) \right\rangle = \begin{bmatrix} .371429l & .052381l^2 & .128571l & -.030952l^2 \\ & .009524l^3 & .030952l^2 & -.007143l^3 \\ & & .371429l & -.052381l^2 \\ \text{symmetric} & & & .009524l^3 \end{bmatrix}$$

$$Q^e_k = \left\langle f, G^e_k(x) \right\rangle = f \begin{bmatrix} .5l \\ .083333l^2 \\ .5l \\ -.083333l^2 \end{bmatrix}$$

The components of N^e_{ijk} associated with the nonlinear response are not in the literature. They are as follows:

$$N^e_{ijk} = \left\langle G^e_i(x)\, G^e_j(x), G^e_k(x) \right\rangle$$

$$N^e_{111} = .307143l$$

$$N^e_{211} = .038492l^2$$

$$N^e_{311} = .064286l$$

$$N^e_{411} = -.017060l^2$$

$$N^e_{221} = .006349l^3$$

$$N^e_{321} = .013889l^2$$

$$N^e_{421} = -.003571l^3$$

$$N^e_{331} = .064286l$$

$$N^e_{431} = -.013889l^2$$

$N^e_{441} = .003175\ell^3$

$N^e_{222} = .001191\ell^4$

$N^e_{322} = .003175\ell^3$

$N^e_{422} = -.000794\ell^4$

$N^e_{332} = .017064\ell^2$

$N^e_{432} = -.003572\ell^3$

$N^e_{442} = .000794\ell^4$

$N^e_{333} = .307143\ell$

$N^e_{433} = -.038492\ell^2$

$N^e_{443} = .006349\ell^3$

$N^e_{444} = -.001190\ell^4$

Notice that, according to Eqn. 23, the N^e_{ijk} coefficients are equal for any permutation of i, j, k (i. e., $N^e_{112} = N^e_{121} = N^e_{211}$, etc.)

D. FORMATION OF SYSTEM MATRICES

The system matrices are obtained from the element matrices as follows:

1. A correspondence table between local and global coordinates is formed
2. The element coefficients of any matrix, say the element stiffness coefficients K^e_{ij}, are substituted directly into the system stiffness matrix K_{IJ} according to the correspondence of element coordinates i, j to system coordinates I, J.[Ref.14]

V. COMPUTER PROGRAMS

Equation (13) is originally of the form:

$$(EIV'')'' - (sV')' + kV^p = q \qquad (29)$$

The following block diagram shows how the investigation proceeds (Figure 3). Equation 13 or 29 will be solved by the Finite Element Method. First consider the case of a classical Winkler foundation, for which the theoretical analysis is readily available. The results from the Finite Element Method will then be compared to the theoretical results. Similarly, the modified Winkler foundation will be solved, also by the Finite Element Method. Since there is no theoretical analysis for the case of these nonlinear problems, some of the results from the Finite Element Method will be compared to those from the Finite Difference Method. Finally, in dealing with continuous foundations, certain types of soils will be considered. Results obtained by varying flexural rigidity of beams, foundation modulus and applied force will be presented in the next chapter.

A. GENERAL PROGRAM ORGANIZATION

The program may be divided into two separate categories, depending on linearity or nonlinearity. The reason for this distinction is that for nonlinear problems, the Brown's iteration technique is applied and double precision is required. In contrast, a simple Gaussian elimination method may be applied to solve the linear systems.

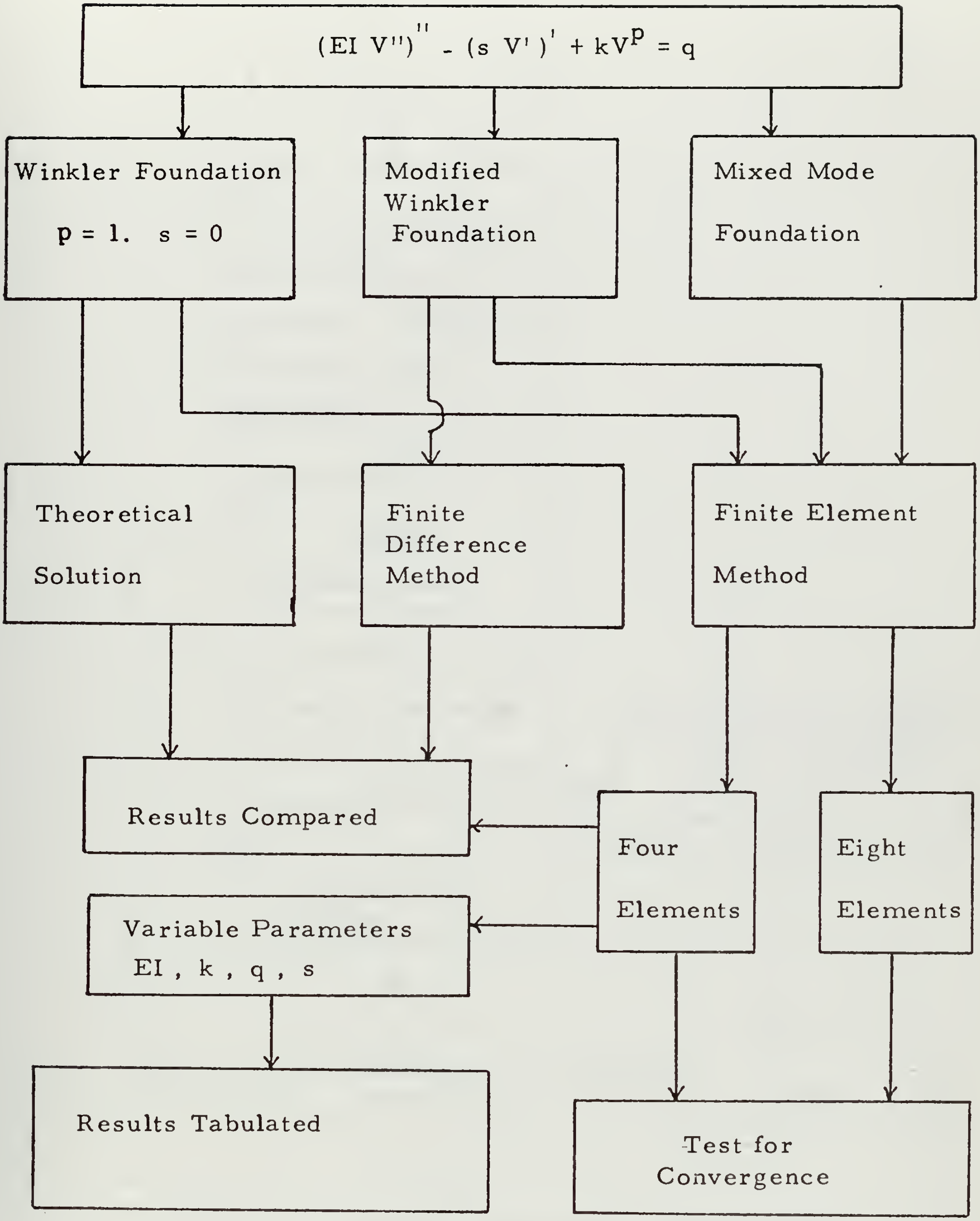


Figure 3. Sequence of the problem investigation

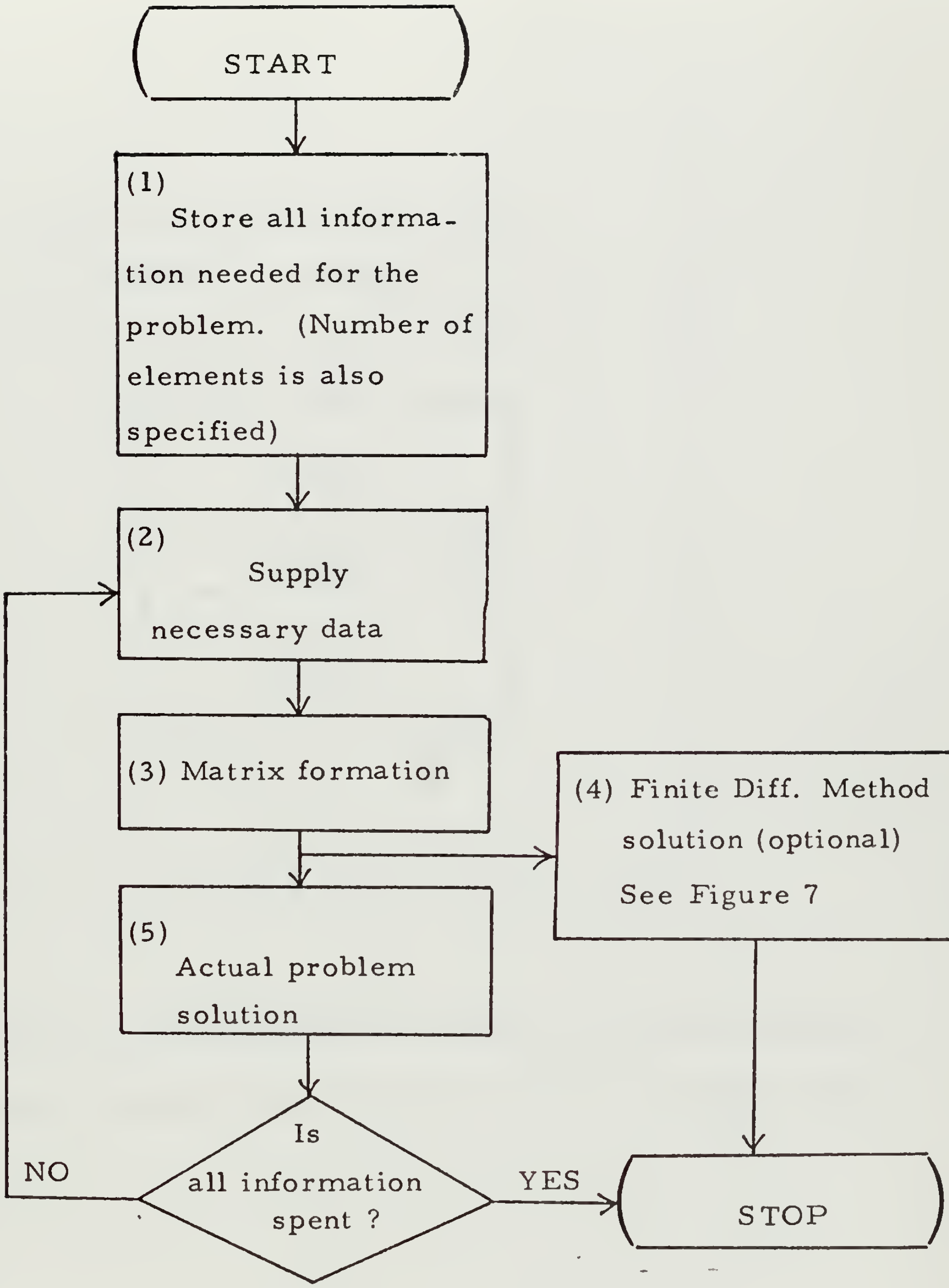


Figure 4. General.organization of the program

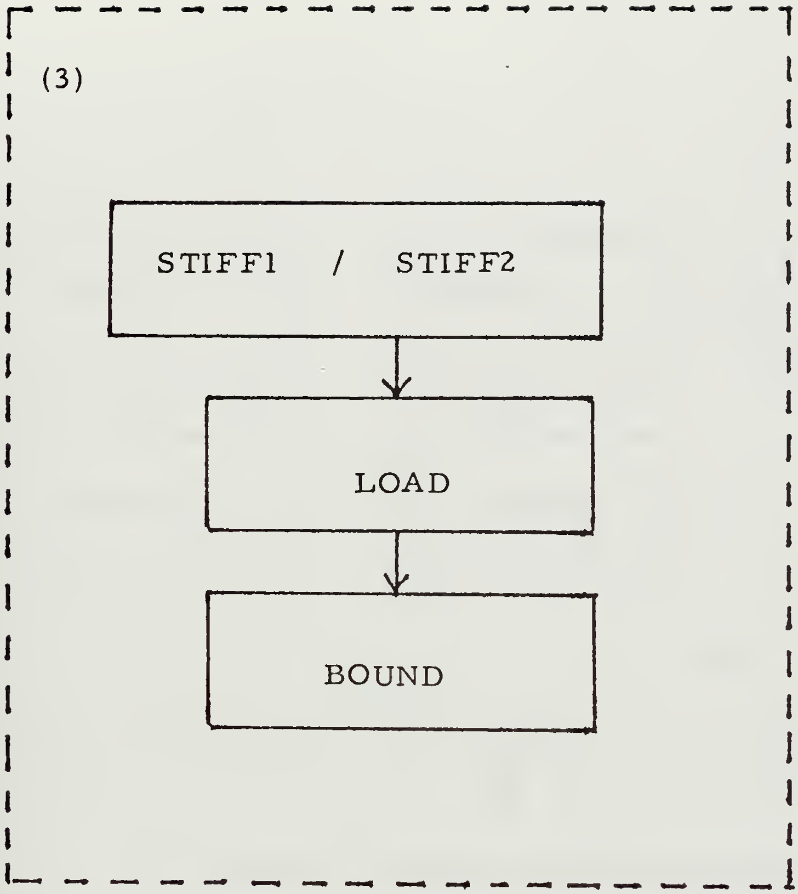


Figure 5. Matrix formation (STIFF1 is used for classical Winkler foundation only, STIFF2 is used for all other cases.)

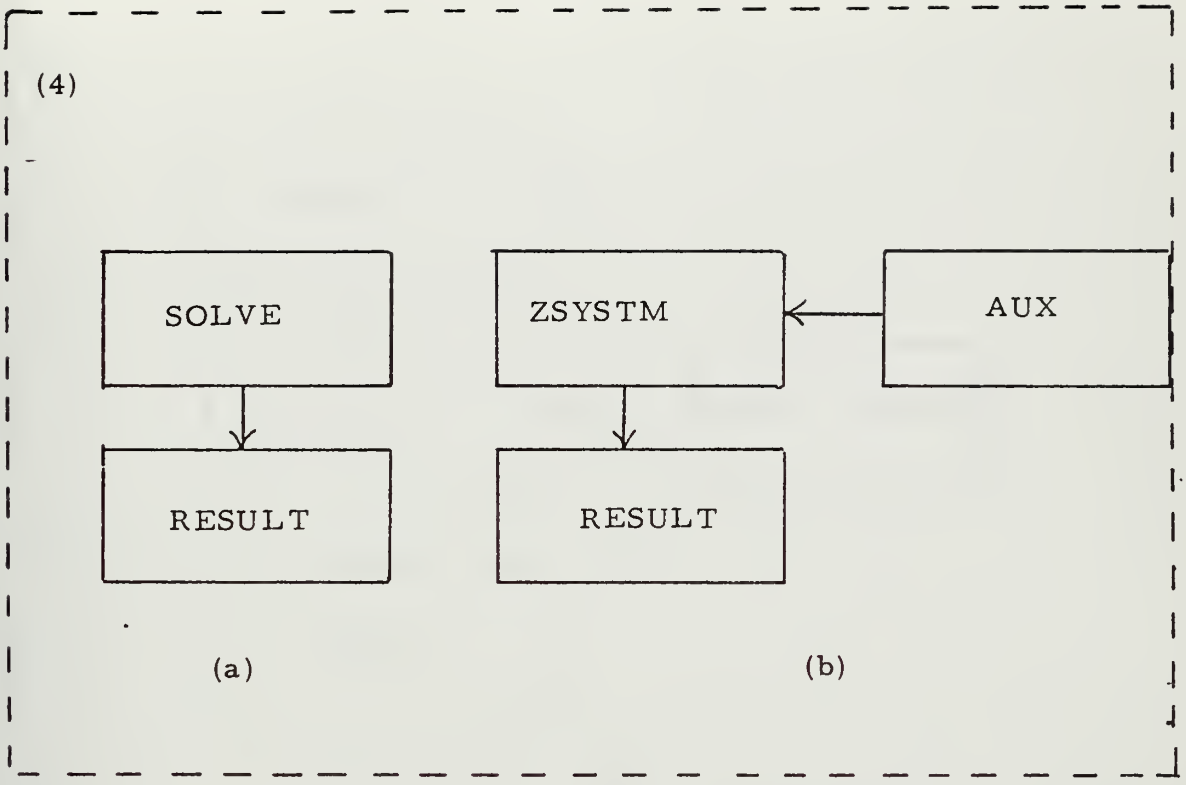


Figure 6. Actual problem solution:

(a) for linear problem,

(b) for nonlinear problem.

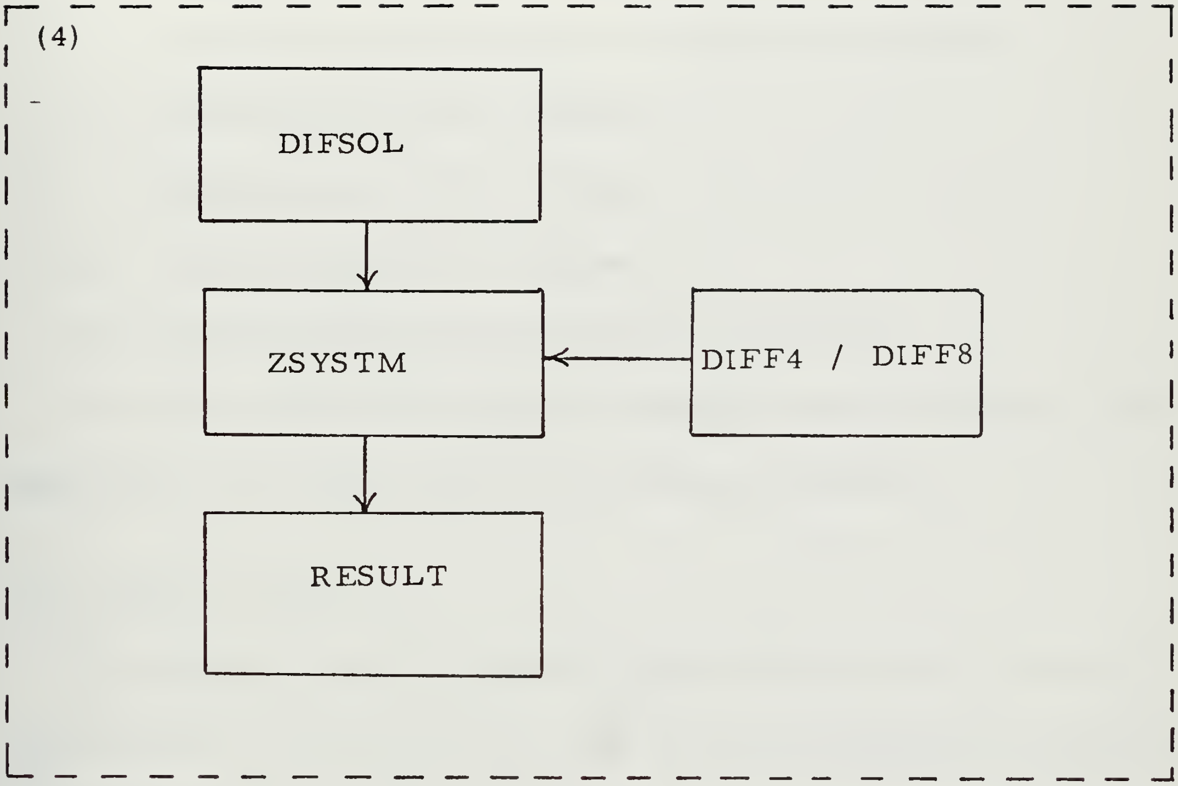


Figure 7. Finite Difference Method solution

Each type of program performs four major functions:

1. Storing all information needed for subsequent problems

2. Supplying necessary data to each particular problem

3.a. Forming the element matrices

 b. Forming the system matrices

4. Solving the algebraic problem.

This organization is shown in Figures 4 through 6.

The Finite Difference Method is applied to some problems to check the results. Its general flow chart is shown in Figure 7.

B. SUBROUTINES

The MAIN program is a control segment that serves to call other subroutines in proper order as required. No calculations are performed in the MAIN program (See Appendix D).

1. <u>Subroutine STORE</u>

This subroutine reads all data available for the problem under investigation, including the maximum number of elements, the number of nodes, the number of boundary conditions, the number of degrees of freedom, the correspondence table of global and local nodes, load conditions, beam properties and foundation characteristics.

2. <u>Subroutine TEST</u>

This subroutine is used particularly for testing the convergence of the solution. In doing so, it selects special information and supplies it to the MAIN program which, keeping all information constant, doubles the number of elements from one run to another.

3. Subroutine CURFIT

For the case of a nonlinear elastic foundation for which the reaction $r(x) = kV^p$, where p is not an integer, a direct application of the finite element method via the Galerkin procedure is not efficient. Using a least square best fit, this subroutine replaces kV^p by a second order polynomial.

4. Subroutine INCHK

Depending on various conditions in changing beam flexural rigidity and foundation modulus, this subroutine selects information from subroutine STORE and supplies it to the MAIN program. This subroutine also prints the appropriate echo check.

5. Subroutines STIFF1 and STIFF2

Subroutine STIFF1 is used for the linear problem and uses single precision; subroutine STIFF2 is used for the nonlinear problem and uses double precision. Both of these subroutines form the element stiffness matrices and assembles the system stiffness matrix.

6. Subroutine LOAD

This subroutine forms the element load vector and assembles the system load vector. It can be used for both linear or nonlinear problems, provided it is in double precision for the later case.

7. Subroutines BOUND1 and BOUND2

These subroutines apply the boundary conditions to the system force vector and the system stiffness matrix produced by subroutines

LOAD, STIFF1 or STIFF2, and transform them into an appropriate form ready to be solved. Subroutine BOUND1 is used for linear problems and subroutine BOUND2 for nonlinear problems.

8. Subroutine SOLVE

This subroutine employs the Gaussian elimination method to solve the linear system of algebraic equations for the linear problem.

9. Subroutine RESULT

This subroutine prints any output data from either linear or nonlinear problems. For nonlinear problems, double precision must be specified.

10. Subroutine DIFSOL

This subroutine solves the nonlinear problem by the Finite Difference Method of which the development will be presented in Chapter V, part D. This solution is optional and was employed in a few cases in order to check the results of the nonlinear problem given by the Finite Element Method.

11. Subroutine FSOIL

Similar to subroutine CURFIT, using a least square fit, this subroutine replaces the total settlement curve of a natural soil given by experimental data [Ref. 30], by a polynomial of second degree. The total settlement curve of the sea floor sediment is supposed to have the same shape as that of the inland soil but its magnitude is reduced by a certain scale.

12. Subroutine VGUESS

This subroutine is designed to give an initial estimate vector to the nonlinear algebraic system before any iteration process is made. It is desirable that if the first estimate has failed to make the iteration convergent, another estimate is provided. In this subroutine, the initial estimate must be taken in the form of a second order polynomial.

13. Function AUX

This is a nonlinear algebraic equation system translated from equation (24), which has to be solved and which serves as an external input to subroutine ZSYSTM.

14. Functions DIFF4 and DIFF8

These are nonlinear algebraic equation systems translated from Eqns. 27 and 28, respectively, which have to be solved by the Finite Difference Method and which serve as external inputs to subroutine DIFSOL.

15. Subroutine LSQPL2

This library subroutine, from the Naval Postgraduate School, Monterey, California, employs a least square best fit method to replace any function by a polynomial.

16. Subroutine ZSYSTM

This library subroutine, from IMSL, solves nonlinear algebraic equations employing the Brown's method of iteration, which is a modified Newton's method and is developed in Reference 13.

C. TESTS FOR CONVERGENCE

The test problems employed here are designed to verify the operation of the computer program and test the convergence of the equation system. The two categories of test performed were linear and nonlinear problems.

An analytic solution and the Finite Difference Method solution were used to verify the Finite Element Method solution for linear and nonlinear cases respectively. The following results (Tables 1, 2, 3) show that for an uniform beam of 100 inches in length, with a uniform load, the 8-element model gives sufficiently accurate results. These analyses show that as the number of elements is increased the finite element method results approach the theoretical results. The small difference between eight and sixteen element results (less than 1%) suggests the solution has converged sufficiently for engineering purposes. For symmetric problems (i.e., symmetric EI, k, s and q), advantage may be taken of symmetry. In this case, only half the system need be considered, thus greatly reducing computer effort. This reduction is especially beneficial for nonlinear problems. The time consumption may increase from 10 to 30 fold in going from a 4-element model to an 8-element model, depending on how good the initial guess is. A brief description of the test program is shown in Figure 8.

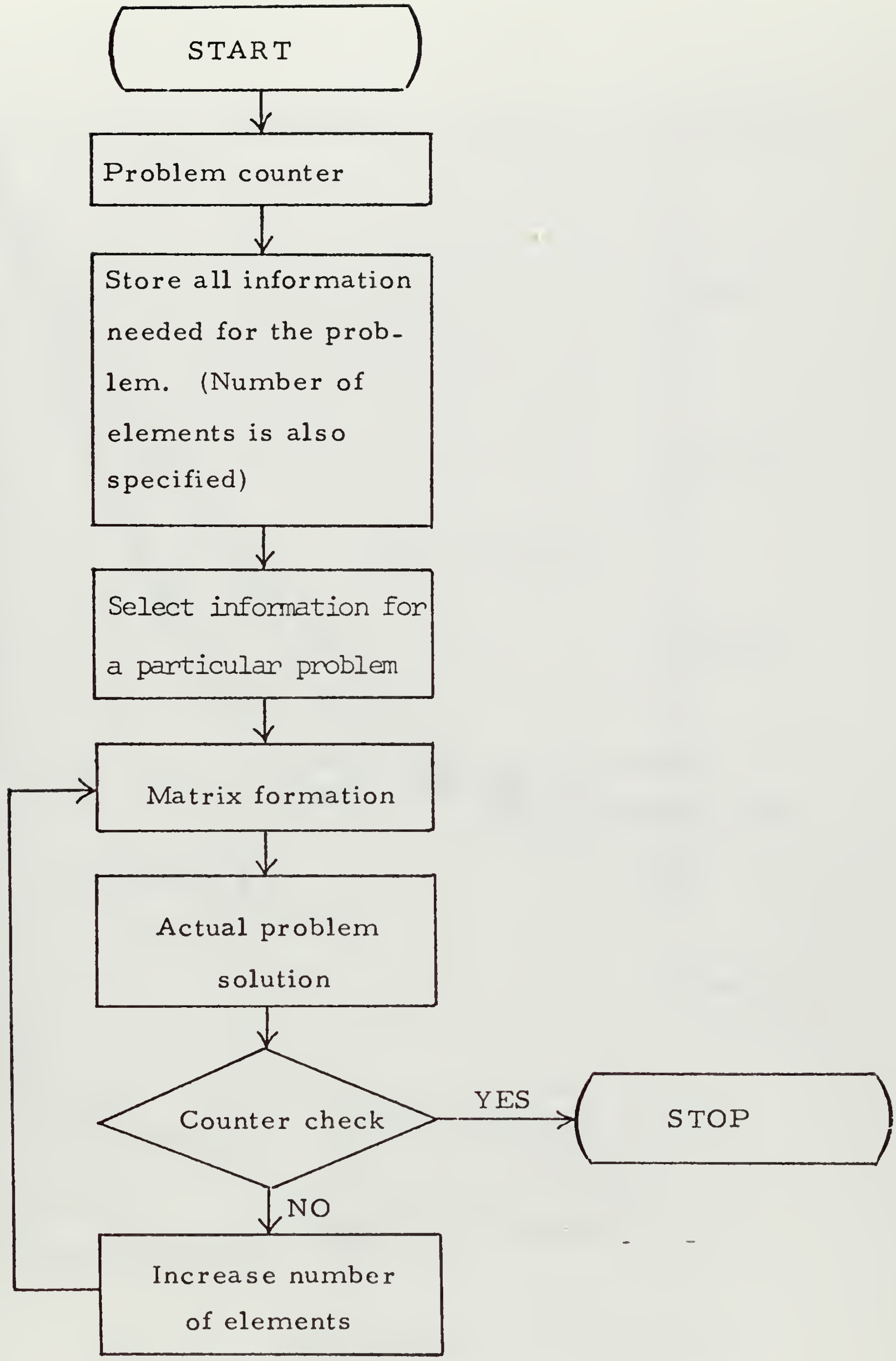


Figure 8. Flow chart of a test program

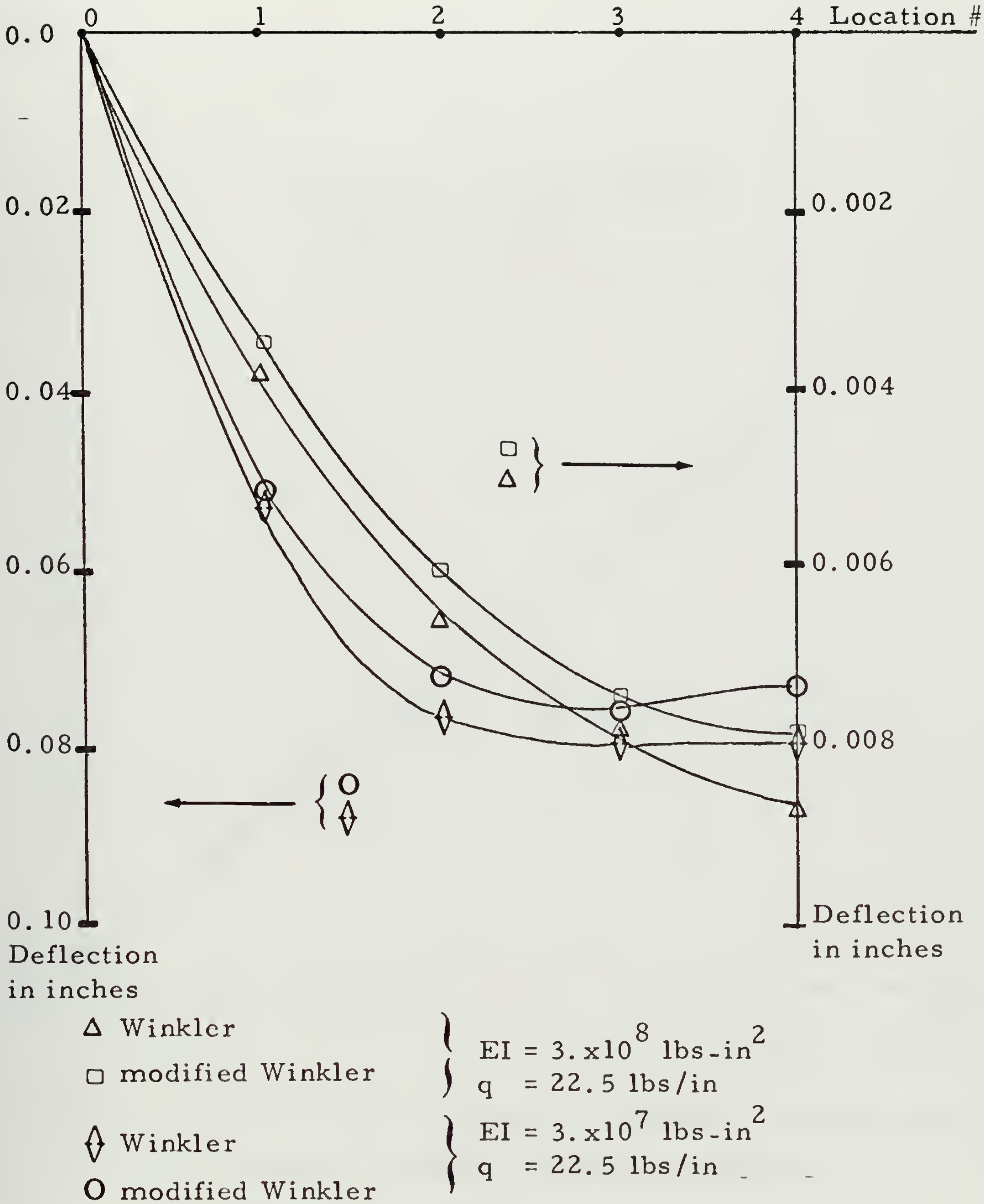


Figure 9. Compared deflection curves of a typical inland soil from Figure 15.

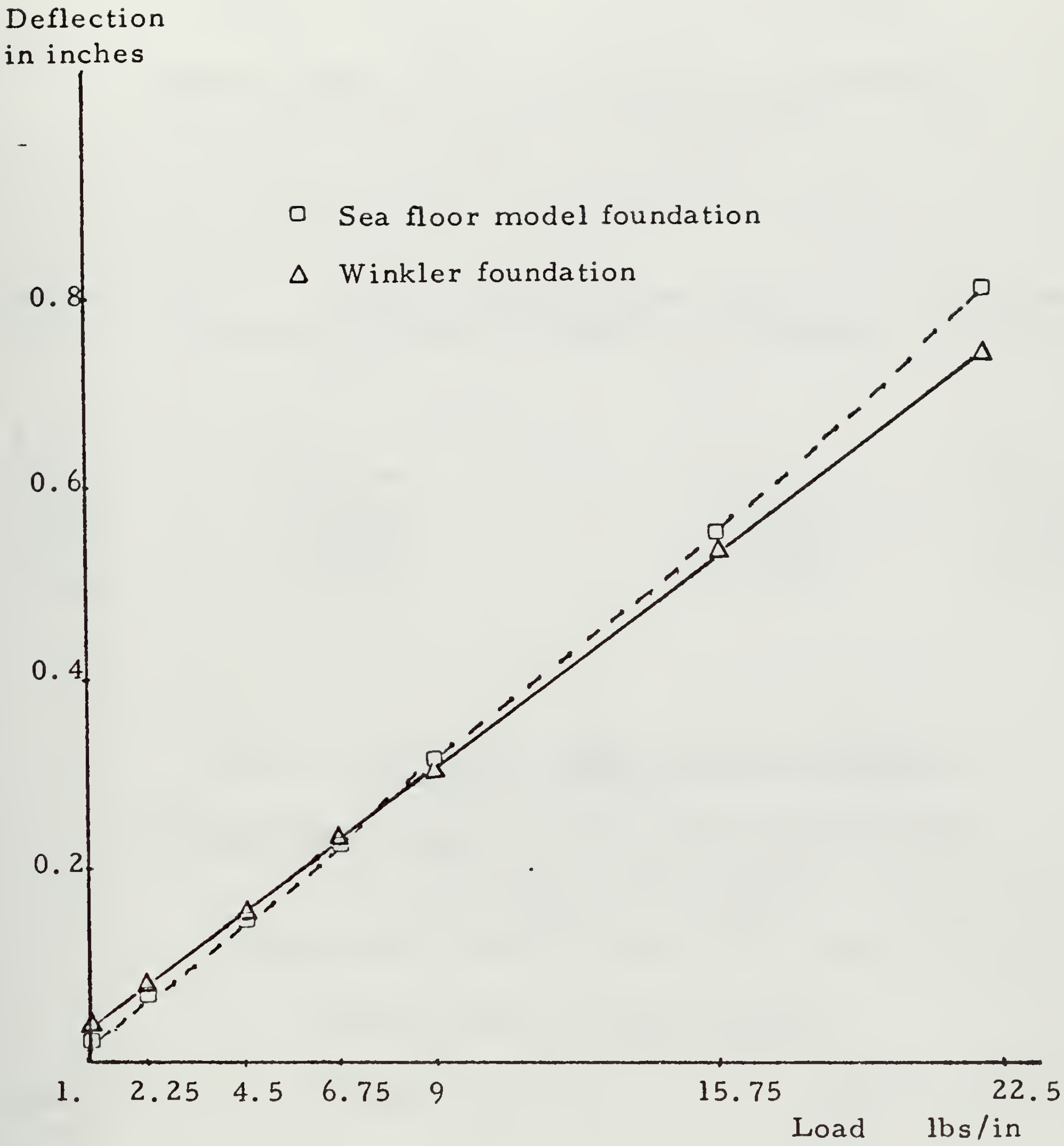


Figure 10. Deflection of a beam on a sea floor model foundation compared to that of Winkler foundation.
(Maximum difference is about 6%)

Table 1. Convergence of the Linear System

$(EIV'')'' + kV = q$

Uniform Simply Supported Beam with Uniform Load

$L = 100$ in, $EI = 3 \times 10^7$ lb-in^2,

$q = 1$ lb/in, $k = 1$ lb/in^2,

Deflection Given in Inches ($\beta L = 0.96$)

Location x/L	Finite Element Method Solution Number of Elements 8	Number of Elements 16	Theoretical Solution
0.0	0.0	0.0	0.0
0.125	0.016296	0.016310	0.016301
0.250	0.029897	0.029919	0.029906
0.375	0.038839	0.038863	0.038880
0.500	0.41950	0.041975	0.042178

Table 2. Convergence of the Nonlinear System

$(EIV'')'' + kV^2 = q$

Uniform Simply Supported Beam with Uniform Load

$L = 100$ in, $EI = 3 \times 10^7$ lb-in^2,

$q = 0.01$ lb/in, $k = 1$ lb/in^2

Deflection Given in Inches ($\beta L = 0.96$)

Finite Element Method Solution:

Location x/L	Number of Elements 4	Number of Elements 8
0.0	0.0	0.0
0.250	0.1451D-03	0.0456D-03
0.500	0.1483D-03	0.1485D-03

Table 3. Solution of $(EIV'')'' + kV^2 = q$
F. E. M. Compared to F. D. M. Uniform
Simply Supported Beam with Uniform Load
$L = 100$ in, $EI = 6.123 \times 10^9$ lb-in^2,
$q = 2.25$ lb/in, Number of Element = 8

Location x/L	k = 100 F. E. M. ITMAX = 3	$\beta L = .80$ F. D. M. ITMAX = 3
0.0	0.0	0.0
0.125	0.185781D-03	0.188397D-03
0.250	0.340909D-03	0.345395D-03
0.375	0.442958D-03	0.448565D-03
0.500	0.478469D-03	0.484450D-03

Location x/L	k = 500 F. E. M. ITMAX = 8	$\beta L = 1.20$ F. D. M. ITMAX = 3
0.0	0.0	0.0
0.125	0.185776D-03	0.188392D-03
0.250	0.340900D-03	0.345385D-03
0.375	0.442946D-03	0.448552D-03
0.500	0.478456D-03	0.484436D-03

Location x/L	k = 1000 F. E. M. ITMAX = 4	$\beta L = 1.42$ F. D. M. ITMAX = 3
0.0	0.0	0.0
0.125	0.185769D-03	0.188385D-03
0.250	0.340889D-03	0.345373D-03
0.375	0.442931D-03	0.448536D-03
0.500	0.478440D-03	0.484419D-03

Location x/L	k = 2000 F. E. M. ITMAX = 4	βL = 1.69 F. D. M. ITMAX = 3
0.0	0.0	0.0
0.125	0.185757D-03	0.188372D-03
0.250	0.340865D-03	0.345349D-03
0.375	0.442900D-03	0.448504D-03
0.500	0.478407D-03	0.484384D-03

Location x/L	k = 3000 F. E. M. ITMAX = 4	βL = 0.87 F. D. M. ITMAX = 4
0.0	0.0	0.0
0.125	0.185745D-03	0.188359D-03
0.250	0.340842D-03	0.345324D-03
0.375	0.442870D-03	0.448472D-03
0.500	0.478374D-03	0.484350D-03

D. THE FINITE DIFFERENCE METHOD

Since there is no analytic solution to the nonlinear case, the Finite Difference Method was used in this thesis to verify the results of the Finite Element Method. The development is restricted to uniform EI and k. The finite difference approximations are developed in Ref. 15. The fourth order derivative has the central difference computational molecule as follows:

$$\left(\frac{d^4V}{dx^4}\right)_j = \frac{1}{l^4}\left[V(j-2) - 4V(j-1)+6V(j)-4V(j+1)+V(j+2)\right] + 0(l^2)$$

where l is an element length

Consider the differential equation of an uniform simply supported beam with uniform load on a uniform nonlinear foundation:

$$EIV'' + kV^2 = q$$

or

$$EIV'' + \frac{k}{(EI)^2}(EIV)^2 = q$$

Let $V^* = EIV$ and $\frac{k}{(EI)^2} = A$, then

the above equation may be rewritten as follows:

$$V^{*''} + A\ V^{*2} - q = 0$$

A computational molecule model applied at node j yields:

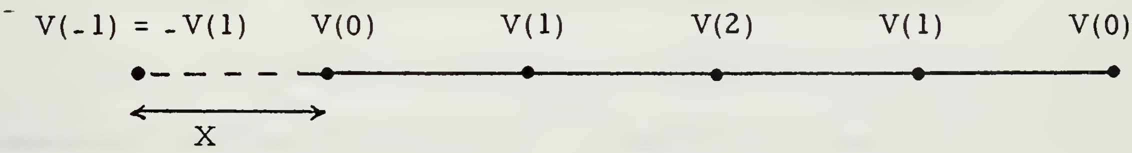
$$\frac{1}{X^4}\quad V^*(j-2) - 4V^*(j-1) + 6V^*(j) - 4V^*(j+1)+V^*(j+2)$$
$$+ A\ V^{*2}(j) - q(j) = 0$$

or:

$$\begin{aligned} &V^*(j-2)-4V^*(j-1) + 6V^*(j) - 4V^*(j+1) + V^*(j+2) \\ &\quad + A\ X^4V^{*2}(j) - X^4q(j) = 0 \end{aligned} \qquad (26)$$

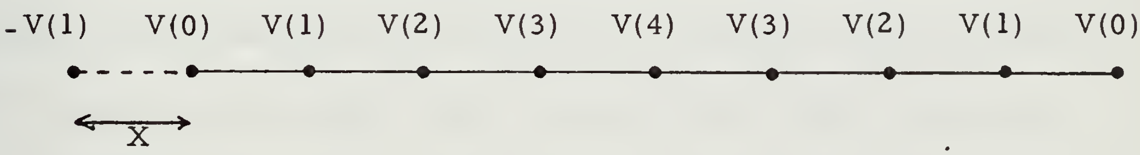
where j ranges from 1 to 2 for 4-element model and from 1 to 4 for 8-element model.

A. FOUR-ELEMENT MODEL



$$\left.\begin{aligned} 6\,V(1) - 4\,V(2) + A\,X^4V^2(1) - qX^4 &= 0 \\ -8\,V(1) + 6\,V(2) + A\,X^4V^2(2) - qX^4 &= 0 \end{aligned}\right\} \qquad (27)$$

B. EIGHT-ELEMENT MODEL



$$\left.\begin{aligned} 5V(1) - 4\,V(2) + V(3) + A\,X^4V^2(1) - qX^4 &= 0 \\ -4V(1) + 6V(2) - 4V(3) + V(4) + A\,X^4V^2(2) - qX^4 &= 0 \\ V(1) - 4V(2) + 7V(3) - 4V(4) + A\,X^4V^2(3) - qX^4 &= 0 \\ 2V(2) - 8V(3) + 6V(4) + A\,X^4V^2(4) - qX^4 &= 0 \end{aligned}\right\} \qquad (28)$$

Notice that the superscript * has been omitted for simplicity; the result V(j) must be multiplied by $(EI)^{-1}$ to get the actual deflection.

The above development is restricted to a uniform beam on a foundation with uniform k. In this case the finite difference method has distinct advantages over the finite element method. In the case of variable EI, k, and s the finite difference method is not easily developed and the advantage shifts to the finite element method.

VI. ILLUSTRATIVE PROBLEMS AND RESULTS

This chapter presents the results of two investigations.

The first investigation attempts to compare the behavior of three foundation models, the Winkler, the modified Winkler and the mixed mode foundations. The comparison is limited to the case of a simply supported uniform beam on foundations with uniform k and s properties. Table 4 gives values for the parameters considered in this study. The value of βL varied from 1.62 to 3.98 i.e., beams of medium length. The results obtained apply only to the particular simply supported systems considered. Other problems may yield other results.

The second investigation seeks to establish that the Finite Element Program developed in this thesis can solve problems with variable EI, k, s and q. As a sample problem, the case of a free-free beam with variable stiffness, variable foundation modulus, and variable load was considered. The βL for this illustrative problem was 3.5.

A. UNIFORM SIMPLY SUPPORTED BEAMS WITH UNIFORM LOADS ON SOIL FOUNDATIONS

Figures 9 and 10 present the deflection of a simply supported beam on soil foundations which are either inland soil or sea floor sediment. The range of parameters considered is shown in Table 4 below.

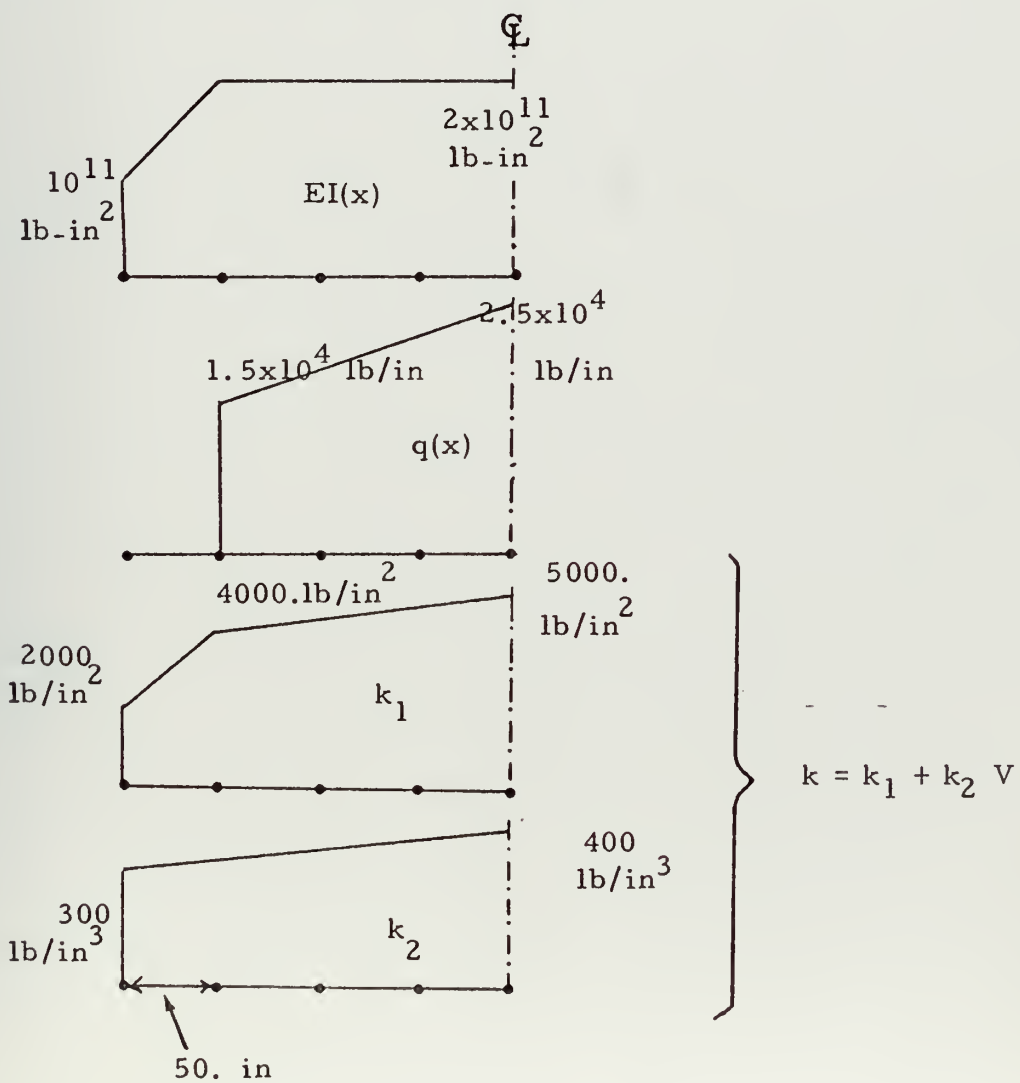
Table 4. Information Data

p	Loading Conditions q (lb/in)	Beam Properties EI ($lb\text{-}in^2$)	Foundation Characteristics			
			Winkler type k(lb/in^2)	Continuous type		Mixed mode (natural soil)
				k lb/in^2	s lb	
2	1	$3x10^7$	1			inland
1.6	2.25	$3x10^8$	50			soil and
1.2	22.5	$6x10^8$	100			sea floor
1.	225.	$9x10^8$	500	8.6	18.3	were
.9			1000	500	1000	chosen
.85			1500			as
.8			2000	3000	6000	examples
			2500			
			3000			

Appendix B contained the tabulated results of Figures 9 and 10 as well as data on the two foundations, inland soil and sea sediment. The results show that the difference between the Winkler hypothesis and other hypotheses is negligible for simply supported beams with uniform properties and loads. This may not be true for beams with other boundary conditions.

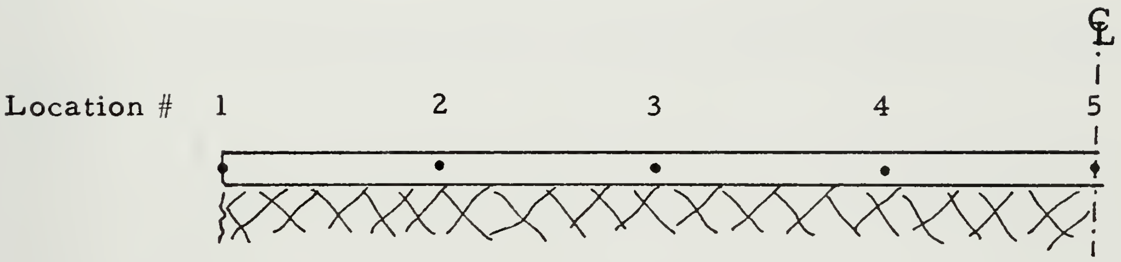
B. A FREE-FREE NON-UNIFORM BEAM PROBLEM

To show how the Finite Element Program can be used for a general case we consider the following problem:



A symmetric problem was considered only to minimize the computer effort. The result is given by Table 5 and Figure 11.

Table 5. Deflection of a Free-Free Non-Uniform Beam



Location #	Deflection, inches
1	1.19150
2	2.09179
3	2.87908
4	3.40052
5	3.58134

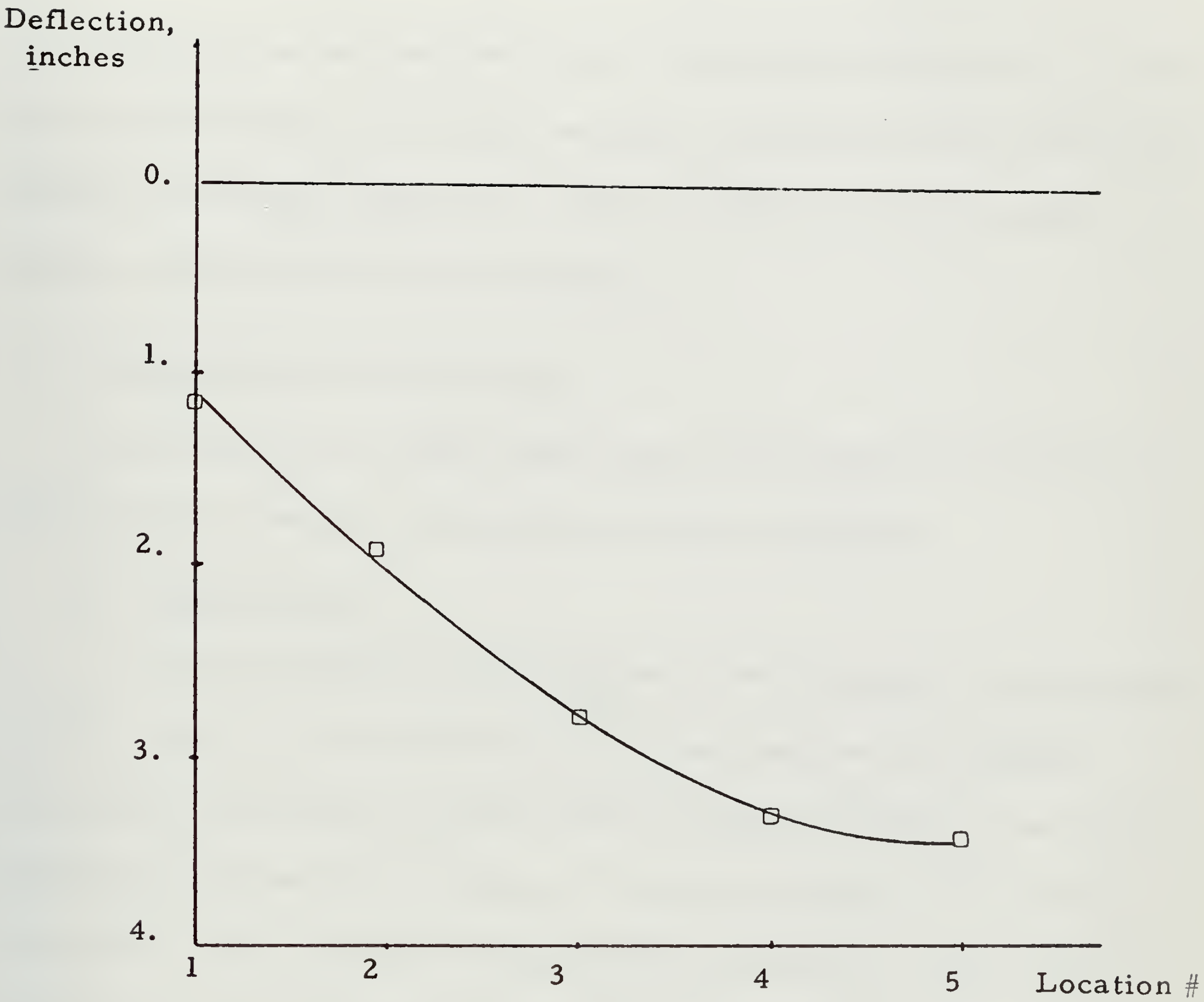


Figure 11. Deflection of a free-free non-uniform beam

VII. CONCLUSIONS AND RECOMMENDATIONS

The computer program presented in this thesis provides an accurate and reliable means for solving a variety of one-dimensional problems of beams on nonlinear foundations. The use of this program and efforts to increase its versatility are encouraged.

A. REMARKS ON THE PROGRAM

The following observations have been made during the development of the program and the investigation of various problems

1. Curve Fitting

It was assumed that the reaction of the foundation is proportional to V^p , where V is the deflection and p any positive real number. Since the Finite Element Method via the Galerkin procedure is not easily applied directly for the case when p is not an interger, V^p has been replaced by a least square fit polynomial of the form $b_1 + b_2V + b_3V^2$. The reaction of the foundation must be zero when there is no deflection, therefore the greatest error may come from the non-zero residual constant b_1 (Figure 12). Suppose the least square fit curve $r_f = b_1 + b_2V + b_3V^2$ intercepts the curve $r_n = V^p$ at the point $V = \overline{V}$. If the deflection of the foundation falls in the vicinity of $\overline{V}$, the result is highly accurate. The result is less accurate, if the deflection is much larger or smaller than $\overline{V}$. One must estimate how large or small the deflection will be, in order to select a "best fit" curve.

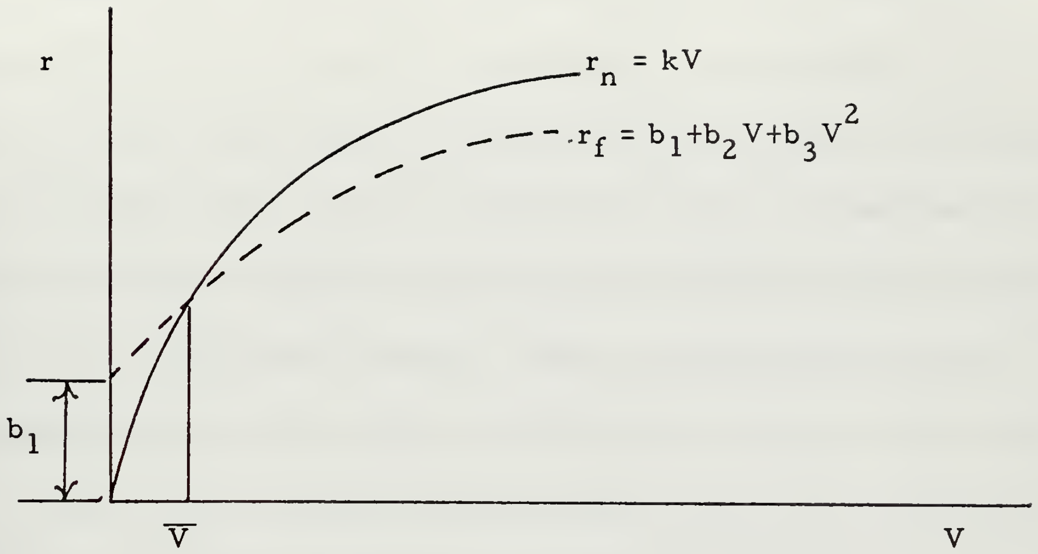


Figure 12. Accuracy of curve fitting

2. Initial Estimate of Deflection

Iterative procedures are among the more popular schemes for solving systems of nonlinear algebraic equations. Such methods require an initial estimate of the solution. In this work the initial estimate was taken in the form of a second order polynomial in the subroutine VGUESS. Not all initial estimates will yield a solution; therefore it is desirable that if one initial estimate fails, it be replaced by another one. This is accomplished in subroutine VGUESS which changes the scale of the initial estimate function.

3. Computational Considerations

In any computer effort there are two major concerns, the space required for a program and the time necessary for solution.

For the linear Finite Element problem the storage area required is proportional to m x M, where m is the number of coordinates and M is the bandwidth of the system. A change in m or M yields a corresponding linear change in the size of the computer storage. In addition the time for solution (for a Gauss elimination type scheme) is of the order of $m \times M^2$ for a symmetric system. These considerations show the disadvantages which result when m or M increase, and are well known.

In the case of nonlinear problems the effects of increasing m and/or M on computational effort are even more pronounced. For example the space requirement associated with the nonlinear terms is $m^2 \times M$. Hence an increase in m from m_1 to m_2 increases the storage in the ratio $\left(\frac{m_2}{m_1}\right)^2$. This is $\frac{m_2}{m_1}$ times greater than in the linear case. Moreover, the computational effort is greatly affected since a large number of terms greatly increases the number of iterations. For example in the solution of an 8-element model (i. e., m = 18, M = 6) non linear problem, the storage for the array is about 8000 bits (single precision) and the computer time was about 60 seconds. When the same problem was modeled by a 4-element model (i. e., m = 10, M = 6), the storage was about 2400 bits (single precision) and the computer time was about 6 seconds; a reduction in storage in the ratio of 3 to 1 and a reduction in computational time in the ratio of 10 to 1.

B. REMARKS ON THE WINKLER HYPOTHESIS

Let $r_n = kV^p$ be the total settlement curve, i.e. the deflection versus reaction curve of a real foundation, which can be obtained by an experiment [Ref. 30]. Let the straight line $r_1 = kV$, where k is the foundation modulus, represent the behavior of the Winkler foundation (Figure 13). Let $\overline{V}$ correspond to the point at which the two curves r_1 and r_n intersect. Consider the area under the curve $r_n = kV^p$ between 0 and V_0. This area is equal to:

$$\int_0^{V_0} r dV = \int_0^{V_0} kV^p \, dV = \frac{k}{p+1} V_0^{p+1} \qquad (30)$$

Consider the functional:

$$F = \int_0^L \left\{ (EIV)''^2 + (\frac{S}{2} V)'^2 + \frac{k}{p+1} V^{p+1} - qV \right\} dx \qquad (31)$$

where L is the length of the beam. The general field equation (29) is the Euler's equation of the above functional. Therefore, to solve the equation (29) is the same as to minimize the functional (31). This functional represents the total potential energy of the system, including the beam, the foundation and the applied load.

The third term under the integral is the energy due to the reaction of the foundation. According to Eqn. 30, the energy (31) is a function of the area under either the r_1 or r_n curve, depending on which foundation is being used. Neglecting the effect due to the shear force in the

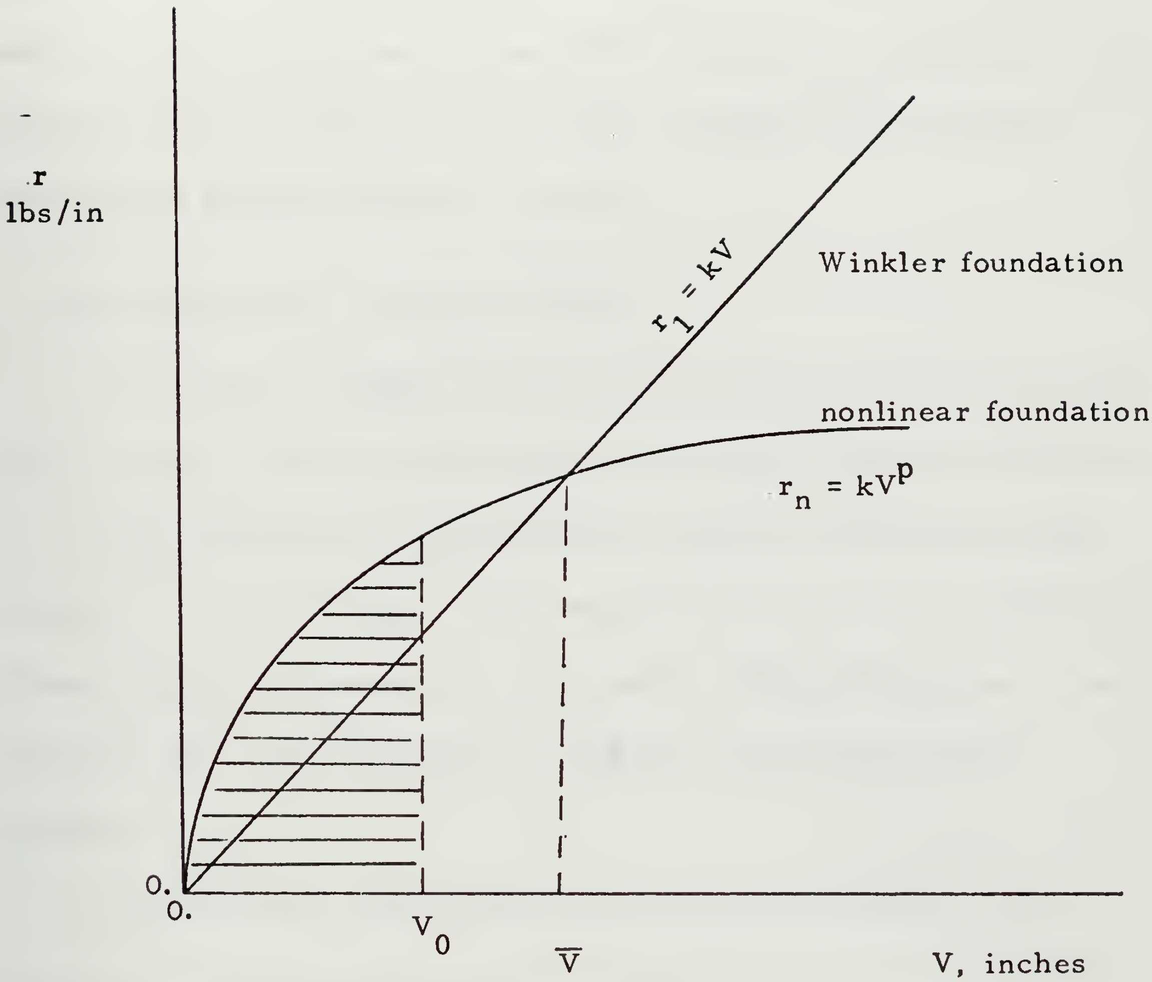


Figure 13. Comparison of a Winkler foundation to other types of foundations

foundation for a while, for a specified beam and a given applied load, one can see that if the deflection is small, say less than $\overline{V}$, the Winkler foundation deflects more than the nonlinear foundation does for equal supporting effort. Now, if the effect of the shear force in the foundation,

i.e., $\frac{s}{2} V'^2$, is taken into account, one can expect that the deflection should be less. The shear force of the foundation depends on $(sV)'^2$ which is usually much smaller than $(EIV)''^2$ and may be neglected. This discussion agrees with the results in two examples of two different types of soils given in Chapter VI, part A.

C. RECOMMENDED FUTURE STUDIES

1. Following a similar procedure, the problem of plates or shells on two- or three- dimensional nonlinear foundations may be investigated.

2. This thesis dealt with nonlinear foundations which are either continuous or discontinuous. Real foundations may be some combination of these foundations; hence there is a need to bridge the gap and consider foundations where the deformation is partly localized and partly continuous.

3. In the present investigation the nonlinear term kV^p was approximated by the second order polynomial, $b_1 + b_2V + b_3V^2$. This restriction results from the finite element approximation of $V(x)$ by $\sum_{i=1}^{m} V_i G_i(x)$. Future studies might consider ways to treat kV^p or in fact the more general case of an arbitrary nonlinear function of V, directly.

APPENDIX A

CONTINUOUS FOUNDATION ANALYSIS

A. GENERAL FORMULATION

This formulation is limited to plane stress theory only. Let u(x, y) and v (x, y) be the displacements at a certain point of the foundation, in the x- and y- directions. In general, an approximate solution can be obtained by expanding u(x, y) and v(x, y) in a finite series:

$$
\left.
\begin{aligned}
u(x, y) &= \sum_{i=1}^{m} U_i(x) \quad \varphi_i(y) \quad , \\
v(x, y) &= \sum_{j=1}^{n} V_j(x) \quad \psi_j(y) \quad ,
\end{aligned}
\right. \qquad (32)
$$

where the dimensionless functions $\varphi_i(y)$ and $\psi_j(y)$ are known and the functions $U_i(x)$ and $V_j(x)$, which have the same dimensions as u(x, y) and v(x, y), are unknown.

From the theory of elasticity, in the two-dimensional plane stress case, the stresses and strains are related as follows [Ref. 3]:

$$
\left.
\begin{aligned}
\sigma_{xx} \quad & \frac{E_f}{1 - \nu_f^2} \left(\epsilon_{xx} + \nu_f \epsilon_{yy} \right) \\
\sigma_{yy} &= \frac{E_f}{1 - \nu_f^2} \left(\nu_f \epsilon_{xx} + \epsilon_{yy} \right) \\
\tau_{xy} &= \tau_{yx} = \frac{E_f}{2(1 + \nu_f)} \gamma_{xy}
\end{aligned}
\right\} \qquad (33)
$$

$$
\left.
\begin{aligned}
\epsilon_{xx} &= \frac{\partial u}{\partial x} \\
\epsilon_{yy} &= \frac{\partial v}{\partial y} \\
\gamma_{xy} &= \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}
\end{aligned}
\right\} \qquad (34)
$$

where E_f and ν_f are the modulus of elasticity and the Poisson's ratio of the foundation, ϵ_{xx} and ϵ_{yy} are the strains in the x- and y- direction and γ_{xy} is the shearing strain.

In terms of the series (32), the relations (33) may be rewritten:

$$
\left.
\begin{aligned}
\sigma_{xx} &= \frac{E_f}{1-\nu_f^2} \left[\sum_{i=1}^{m} U_i' \varphi_i + \nu \sum_{j=1}^{n} V_j \psi_j' \right] \\
\sigma_{yy} &= \frac{E_f}{1-\nu_f^2} \left[\nu \sum_{i=1}^{m} U_i' \varphi_i + \sum_{j=1}^{n} V_j \psi_j' \right] \\
\tau_{xy} &= \tau_{yx} = \frac{E_f}{2(1+\nu_f)} \left[\sum_{i=1}^{m} U_i \varphi_i' + \sum_{j=1}^{n} V_j' \psi_j \right]
\end{aligned}
\right\} \qquad (35)
$$

where the "prime" denotes the derivative of the function with respect to its own argument.

From the foundation model shown in Figure 1, consider now an elementary strip of length Δx (Figure 14). The functions $U_i(x)$ and $V_j(x)$ can be obtained from the equilibrium conditions. According to

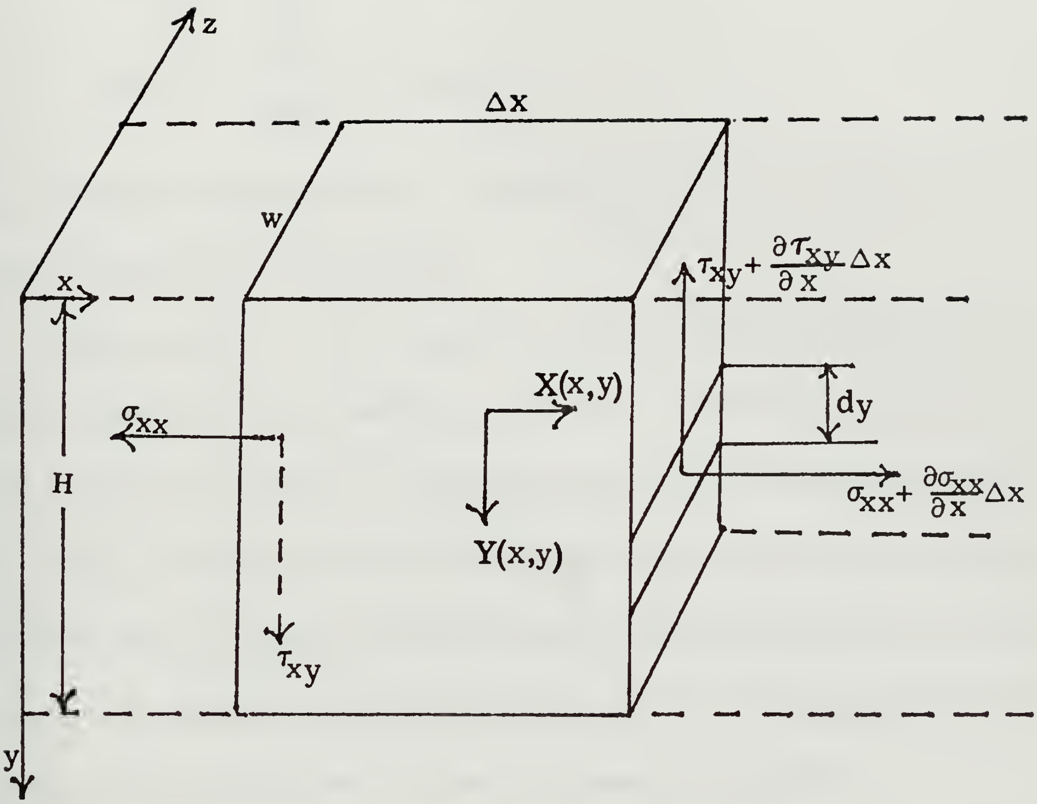


Figure 14. Force system on a strip of the foundation

Lagrange's principle of virtual displacements,[1] the total work of all internal and external forces acting on this strip due to any virtual displacement is zero. Let:

$$\bar{u}(x, y) = \sum_{i=1}^{m} \bar{U}_i \; \varphi_i(y) \quad , \tag{36}$$

$$\bar{v}(x, y) = \sum_{j=1}^{n} \bar{V}_j \; \psi_j(y) \quad , \tag{37}$$

be the virtual displacements, where $\bar{U}_i$ and $\bar{V}_j$ are arbitrary constants at that location x of the strip.

Since there is no force applied on the surface of the foundation, and the bottom of the foundation rests on a rigid base, there is no work done on these two faces. It is assumed that all properties of the foundation remain constant in the z-direction, hence there is no pressure gradient in the z-direction and therefore the total work done on two faces perpendicular to the z-direction is zero. The system of forces which must be taken into account is shown in Figure 14.

The external forces are the result of normal stresses σ_{xx}, $\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} \Delta x$, the shearing stresses τ_{xy}, $\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} \Delta x$ and the distributed forces, $X(x, y)$ in the x-direction and $Y(x, y)$ in the Y-direction.

The internal forces are the result of normal stresses σ_{yy} and shearing stresses τ_{xy}. The work done by these internal forces is

---

[1] A virtual displacement is any displacement which is consistent with the forces and constraints [Ref. 2].

the strain energy of the strip [Refs. 4, 9].

The total work done on the strip by any m + n virtual displacements is then given by the following expression:

$$- \int_0^H \left(\frac{\partial \tau_{xx}}{\partial x} \Delta x \right) (wdy) \; \bar{u}_i \; - \; \int_0^H (\tau_{xy} \bar{\gamma}_{xy}) (\Delta x wdy)$$

$$+ \int_0^H X \; (x, y) \; (\Delta \, xwdy) \; \bar{u}_i \; + \; \int_0^H \left(\frac{\partial \tau_{xy}}{\partial x} \Delta x \right) (wdy) \; \bar{v}_j \tag{38}$$

$$- \int_0^H (\sigma_{yy} \; \bar{\epsilon}_{yy}) \; (\Delta x \; \; wdy) + \int_0^H Y \; (x, y) \; (\; \Delta x \; \; wdy \;) \; \; \bar{v}_j \; = \; 0$$

$(i = 1, 2, 3, \ldots, m \; ; \; j = 1, 2, 3, \ldots, n)$

where

$$\bar{\gamma}_{xy} = \frac{\partial \bar{u}_i}{\partial y} + \frac{\partial \bar{v}_j}{\partial x} = \bar{U}_i \; \varphi'(y) \tag{39}$$

$$\bar{\epsilon}_{yy} = \frac{\partial \bar{v}_j}{\partial y} = \bar{V}_j \; \psi'(y) \tag{40}$$

$\int_0^H \left(\frac{\partial \tau_{xx}}{\partial x} \Delta x\right) (wdy) \; \bar{u}_i$ = work of external forces $\left(\frac{\partial \tau_{xx}}{\partial x} \Delta x\right)(wdy)$;

$\int_0^H (\tau_{xy} \; \bar{\gamma}_{xy}) \; (\Delta x \; \; wdy)$ = strain energy due to shearing stress τ_{xy};

$\int_0^H \left(\frac{\partial \tau_{xy}}{\partial x} \Delta x\right) (wdy) \; \bar{u}_j$ = work of external forces $\left(\frac{\partial \tau_{xy}}{\partial x} \Delta x \right) (wdy)$;

$\int_0^H (\sigma_{yy} \; \bar{\epsilon}_{yy}) \; (\Delta x \; \; wdy)$ = strain energy due to normal stress σ_{yy}

$\int_0^H X \, (x, y)(\Delta x \; wdy) \; \bar{u}_i + \int_0^H Y(x, y) \; (\Delta x \; wdy) \; \bar{v}_j$ = work of body forces.

After dividing through by $w\Delta x$ and substituting $\overline{u}_i$, $\overline{v}_j$, $\overline{\gamma}_{xy}$, $\overline{\epsilon}_{yy}$ by (36), (37), (39) and (40), the expression (38) may be rewritten as follows:

$$\sum_{i=1}^{m}\left[\int_0^H \frac{\partial \sigma_{xx}}{\partial x}\,\overline{U}_i\,\varphi_i(y)\,dy - \int_0^H \tau_{xy}\,\overline{U}_i\,\varphi_i'(y)\,dy + \int_0^H X(x,y)\,\overline{U}_i(y)dy\right]$$

$$+\sum_{j=1}^{n}\left[\int_0^H \frac{\partial \tau_{xy}}{\partial x}\,\overline{V}_j\,\psi_j(y)dy - \int_0^H \sigma_{yy}\overline{V}_j\,\psi_j'(y)\,dy)\right.$$

$$\left. + \int_0^H Y\,(x,y)\,\overline{V}_j\,\psi_j(y)\,dy\right] = 0 \qquad (41)$$

or

$$\sum_{i=1}^{m}\left\{\overline{U}_i\left[\int_0^H \frac{\partial \sigma_{xx}}{\partial x}\varphi_i(y)\,dy - \int_0^H \tau_{xy}\varphi_i'(y)\,dy + \int_0^H X(x,y)\varphi_i(y)dy\right]\right\}$$

$$+\sum_{j=1}^{n}\left\{\overline{V}_j\left[\int_0^H \frac{\partial \tau_{xy}}{\partial x}\psi_j(y)\,dy - \int_0^H \sigma_{yy}\,\psi_j'(y)\,dy + \int_0^H Y(x,y)\psi_j(y)dy\right]\right\}$$

(42)

Since $\overline{U}_i$ and $\overline{V}_j$ are arbitrary, the above expression leads to the two equations in the x- and y-directions:

$$\int_0^H \frac{\partial \sigma_{xx}}{\partial x}\varphi_i(y)dy - \int_0^H \tau_{xy}\varphi_i'(y)dy + \int_0^H X(x,y)\,\varphi_i(y)dy = 0 \qquad (43)$$

$$\int_0^H \frac{\partial \tau_{xy}}{\partial x}\psi_j(y)dy - \int_0^H \sigma_{yy}\,\psi_j'(y)dy + \int_0^H Y(x,y)\,\psi_j(y)dy = 0 \qquad (44)$$

Substitution of (35) into (41) and (42) leads to a system of m + n second order differential equations in $\bar{U}_i(x)$ and $\bar{V}_j(x)$ as follows:

$$\left.\begin{aligned}
&\int_0^H \frac{\partial}{\partial x} \frac{E_f}{1-\nu_f^2} \left[\sum_{i=1}^{m} \bar{U}_i' \varphi_i + \nu \sum_{j=1}^{m} \bar{V}_j \psi'_j \right] \varphi_k \, dy \\
&- \int_0^H \frac{E_f}{2(1+\nu_f)} \left[\sum_{i=1}^{m} \bar{U}_i \varphi_i' + \sum_{j=1}^{n} \bar{V}_j' \psi_j \right] \varphi_k' \, dy + \int_0^H X \varphi_k \, dy = 0 \\
&\int_0^H \frac{\partial}{\partial x} \frac{E_f}{2(1+\nu_f)} \left[\sum_{i=1}^{m} \bar{U}_i \varphi_i' + \sum_{j=1}^{n} \bar{V}_j' \psi_j \right] \psi_\ell \, dy \\
&- \int_0^H \frac{E_f}{1-\nu_f^2} \left[\nu \sum_{i=1}^{m} \bar{U}_i' \varphi_i + \sum_{j=1}^{n} \bar{V}_j \psi_j' \right] \psi_\ell' \, dy + \int_0^H Y \psi_\ell \, dy = 0
\end{aligned}\right\} \quad (45)$$

$$(k = 1, 2, 3, \ldots m$$
$$\ell = 1, 2, 3, \ldots, n)$$

These are the two governing equations of a two-dimensional foundation which has a depth H.

B. SPECIAL CASE

For the limited scope of this thesis, let the horizontal displacements u be negligible. This means that any virtual displacements in the x-direction are also negligible. Since $\overline{U_i}$ are arbitrary constants, Eqn. 36 implies that $\varphi_i(y)$ must be identically zero. Only the last equation of the system (45) remains and it reduces to:

$$\sum_{i=1}^{m}\left\{ \frac{\partial}{\partial x} \frac{E_f}{2(1+\nu_f)} \int_0^H V_i' \Psi_i \Psi_j \, dy - \frac{E_f}{1-\nu_f^2} \int_0^H V_i \Psi_i' \Psi_j \, dy \right\}$$

$$+ \int_0^H Y(x,y) \, \psi_j \, dy = 0 \tag{46}$$

where j = 1, 2, 3, . . . , n, and E_f and V_f are assumed to be constant.

The above expression may be rewritten:

$$\sum_{i=1}^{m}\left\{ \frac{\partial}{\partial x} \left[\left(\frac{E_f}{2(1+\nu_f)} \int_0^H \psi_i \psi_j \, dy\right) V_i' \right] - \left[\frac{E_f}{1-\nu_f^2} \int_0^H \psi_i' \psi_j' \, dy \right] V_i \right\}$$

$$+ \int_0^H Y(x,y) \, \psi_j \, dy = 0 \tag{47}$$

or

$$r_j = \sum_{i=1}^{m} \left\{ (s_{ij} V_i')' - k_{ij} V_i \right\} \tag{48}$$

$$j = 1, 2, 3, \ldots, n$$

where

$$r_j = \int_0^H Y(x,y) \, \psi_j \, dy \tag{49}$$

is the reaction of the foundation.

$$s_{ij} = \frac{E_f}{2(1+\nu_f)} \int_0^H \psi_i \, \psi_j \, dy \tag{50}$$

$$k_{ij} = \frac{E_f}{1-\nu_f^2} \int_0^H \psi_i' \, \psi_j' \, dy \tag{51}$$

The coefficients s_{ij}, lbs/in, and k_{ij}, lbs/in^3, are the two characteristics of a foundation [Ref. 5]. The former accounts for the shearing stress distribution in the foundation material, and the latter is equivalent to the Winkler constant of proportionality. They are however, dependent on how the ψ_i functions are chosen and are consequently not independent of each other.

For simplicity, assume that the foundation has an infinite depth, which is usually true for any soil layer compared to the deflection of most structural systems. Let the one-dimensional function $\psi(y)$ satisfy the boundary conditions:

$$\psi(o) = 1. \tag{52}$$

$$\psi(\infty) = 0. \tag{53}$$

Then the displacement at any point of the foundation is:

$$v(x, y) = V(x)\ \psi(y) \tag{54}$$

in which V(x) is the deflection at the surface of the foundation, where y = 0. Furthermore, assume that the displacements decrease exponentially with the depth of the foundation, then $\psi(y)$ may be expressed as:

$$\psi(y) = e^{-\lambda y}. \tag{55}$$

where λ is a known constant which depends on the properties of the foundation. The values of s and k^* are then given by (50) and (51):

$$s = \frac{E_f}{2(1+\nu_f)} \int_0^\infty e^{-2\lambda y} dy = \frac{E_f}{4\lambda(1+\nu_f)} \quad \text{lbs/in} \tag{56}$$

$$k^* = \frac{E_f}{1-\nu_f^2} \int_0^\infty (e^{-\lambda y})'^2 dy = \frac{\lambda E_f}{2(1-\nu_f^2)} \quad \text{lbs}/_{\text{in}}{}^3 \tag{57}$$

Equation (48) may be rewritten

$$r = (sV')' - k^*V \tag{58}$$

where s and k^* are given by (56) and (57).

C. NUMERICAL APPLICATIONS

In accordance with soil mechanics, taking the length of a loaded area as unity, Reference 7 gives an empirical relation between bearing load and settlement of a foundation as follows:

$$r = \frac{1}{\mu} \frac{E_f}{1-\nu_f} V \tag{59}$$

where μ is called the influence coefficient, which depends on the shape of loaded area and the properties of the foundation. The values of μ are given in Table 6. [Ref. 6]. Comparison of the Winkler hypothesis

$$r = kV \tag{60}$$

to equation (59) yields:

$$k = \frac{1}{\mu} \frac{E_f}{1-\nu_f^2} \tag{61}$$

Eqns. 57 and 61 show the similarity of the theoretical analysis and the empirical result. The value λ can be computed by equating (57) to (61):

$$\lambda = 2\mu \tag{62}$$

Table 6
Influence Coefficients

(After Z. Wilun and K. Starzewski [Ref. 7])

Shape of foundation	Flexible foundation			Rigid foundation
	Settlement of centre	Settlement of corner	Mean Settlement	Settlement
	μ_0	μ_c	μ	μ_r
Circular	1.00	0.64 point on perimeter	0.85	0.79
Square	1.12		0.95	0.88
Rectangle		$\mu_c = \frac{1}{2}\mu_o$		
2 x 1	1.53		1.30	1.22
5 x 1	2.10		1.83	1.72
10 x 1	2.53		2.25	2.12
100 x 1	4.00		3.69	--

For the purpose of this thesis, consider a loaded area of shape $w \times \Delta x$, where Δx is taken as a unit length, one inch, and w is equal to the width of the foundation (Figure 14). Assume that w ranges between 5 inches and 10 inches. From Table 6, $1.83 < \mu < 2.25$. Let $\mu = 2$, then, from (62) $\lambda = 4$. In order to determine s and k, the modulus of elasticity and the Poisson's ratio of the foundation must be known.

The modulus of elasticity of rocks ranges from 100 to 10,000 mn/m^2 ($7 mn/m^2 \cong 1 klbf/in^2$) and the Poisson's ratio from 0.1 to 0.3 [Refs. 10, 7]. Deep sea sediment cores have modulii of elasticity ranging from 0 to 12.4 psi [Ref. 11]. The modulii of elasticity of beach soils vary from 30 to 570 psi, while those of inland soils vary from 3,000 psi to 11,500 psi [Ref. 7]. Most soils have a Poisson's ratio near to 0.5 [Ref. 12]. The values of s and k computed from (56) and (57) for various types of soils are given in Table 7.

Table 7

s and k Values Depending on Various Types of Soils (Loaded Area of Shape $\Delta x = 1''$, $5'' < w < 10''$)

	Sea Floor Sediment	Beach Soils	General Inland Soils
E_f, psi	12.4	30-570	3000-11,500
ν_f	0.5	0.5	.45
s, lbs	18.6	50	5,000-21,000
k, psi	8.3	20-380	1800-7200

APPENDIX B

DATA AND RESULTS FOR PROBLEMS OF CHAPTER VI, PART A

The following example refers to Figure 23-10 of Reference 30. Assume that the piles used for this soil have 24-inch diameters. The reaction for the foundation is

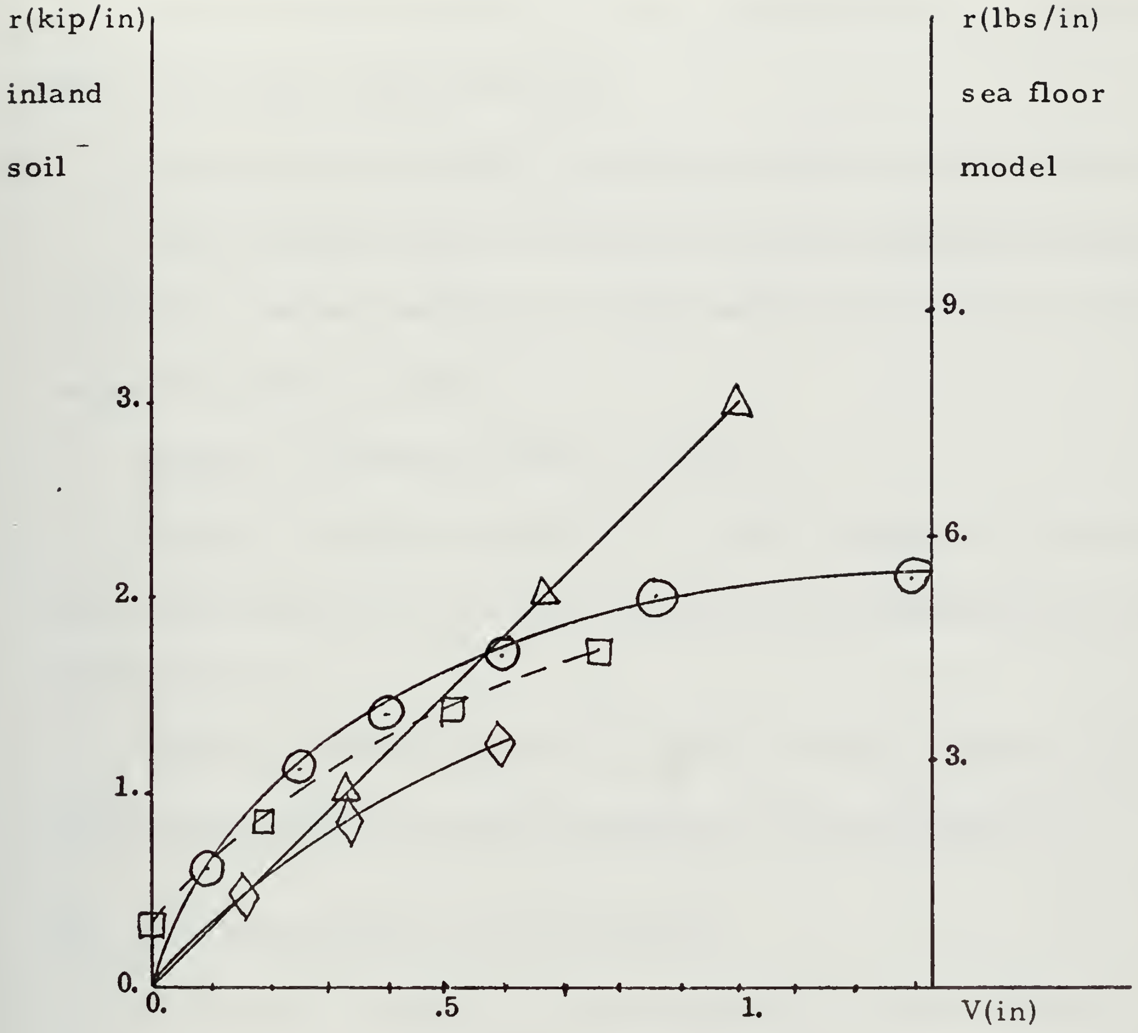
$$r = \frac{(\text{Test load})\ w}{\pi D^2/4} \quad \text{lbs/in}$$

With D = 24 inches and w = 6.5 inches we have the following table.

Table 8
Deflection Versus Reaction of a Natural Soil

From Experimental Data [Ref. 30] Deflection (inches)	Test load (tons)	lbs/in
0.10	20.	574.6
0.25	40.	1149.2
0.40	50.	1436.5
0.60	60.	1724.5
0.90	70.	2011.8
1.30	73.	2097.6

A curve of this data is shown in Figure 15.



⊙ Experimental data curve

□ Least square fit

◇ Assumed foundation behavior curve

△ Approximate Winkler hypothesis, $k = 3000.\ lbs/in^2$

Figure 15. Foundation reaction versus deflection

(After Spangler, M. G., [Ref. 30])

A. A TYPICAL INLAND SOIL

Figure 15 shows reactions versus displacements of a typical inland soil (read scale on left hand side).

Assume that the Winkler foundation modulus is about 3000 lb/in^2.

The deflection results, obtained by the Finite Element program, of a simply supported uniform beam with uniform load resting on this type of soil are given by Table 9.

Column 1: Solution of $(EIV'')'' + kV = q$

Column 2: Solution of $(EIV'')'' + r(V) = q$ where $r(V)$ is replaced by a least square fit curve of the reaction versus deflection curve given in Figure 15.

Column 3: Solution of $(EIV'')'' - (sV')' + r(V) = q$ where s and k corresponding to a particular foundation is given by Table 7.

B. AN ASSUMED SEA FLOOR MODEL

Since there is insufficient experimental data of sea sediment, it will be assumed that the reaction versus deflection curve of sea sediment has the same shape as that of inland soil. According to Table 4 and Reference 11, assume that the Winkler foundation modulus of sea floor is about 8.3 lb/in^2. It will be assumed that the inland soil reaction versus deflection curve is scaled down by 8.3/3000 to obtain the sea floor reaction versus deflection curve. (Figure 15. read right hand side scale).

The deflection results of a simply supported uniform beam with uniform load resting on this type of sea foundation model is given by Table 10.

Table 9. Compared Foundation Deflection of Various Hypotheses for a Typical Inland Soil Represented in Figure 15

a) L = 100 in, EI = 3×10^8 lbs-in^2, q = 22.5 lbs/in
k = 3000 lbs/in^2, s = 6000 lbs/in

Location x/L	(1) Winkler	(2) Modified Winkler	(3) Mixed mode foundation
0.0	0.0	0.0	0.0
0.125	0.003690	0.350190D-02	0.349588D-02
0.250	0.006390	0.603699D-02	0.602663D-02
0.325	0.007898	0.744817D-02	0.743554D-02
0.500	0.008370	0.788845D-02	0.787515D-02

b) L = 100 in, EI = 3×10^7 lbs-in^2, q = 22.5 lbs/in
k = 3000 lbs/in^2, s = 6000 lbs/in

Location x/L	(1) Winkler	(2) Modified Winkler	(3) Mixed mode foundation
0.0	0.0	0.0	0.0
0.125	0.005513	0.524425D-02	0.522121D-02
0.250	0.007740	0.728582D-02	0.726411D-02
0.375	0.007988	0.749146D-02	0.748313D-02
0.500	0.008010	0.739400D-02	0.739234D-02

c) L = 100 in, EI = 3×10^7 lbs-in^2, q = 225 lbs/in
k = 3000 lbs/in^2, s = 6000 lbs/in

Location x/L	(1) Winkler	(2) Modified Winkler	(3) Mixed mode foundation
0.0	0.0	0.0	0.0
0.125	0.055125	0.536074D-01	0.533658D-01
0.250	0.077400	0.749189D-01	0.746865D-01
0.375	0.079875	0.773826D-01	0.772876D-01
0.500	0.080100	0.764925D-01	0.764673D-01

Table 10. Compared Foundation Deflection of Various Hypotheses for a Sea Floor Soil Represented by Figure 15

L = 100 in, EI = 3×10^7 lbs-in^2

a) q = 1 lbs/in

Location x/L	Winkler	Modified Winkler	Mixed mode foundation
0.0	0.0	0.0	0.0
0.125	0.131617E-01	0.130091D-01	0.130029D-01
0.250	0.241054E-01	0.238239D-01	0.238124D-01
0.375	0.312727E-01	0.309054D-01	0.308904D-01
0.500	0.337603E-01	0.333630D-01	0.333467D-01

b) q = 2.25 lbs/in

Location x/L	Winkler	Modified Winkler	Mixed mode foundation
0.0	0.0	0.0	0.0
0.125	0.296138E-01	0.293785D-01	0.293644D-01
0.250	0.542371E-01	0.538043D-01	0.537781D-01
0.375	0.703635E-01	0.698002D-01	0.697660D-01
0.500	0.759606E-01	0.753519D-01	0.753150D-01

c) q = 4.5 lbs/in

Location x/L	Winkler	Modified Winkler	Mixed mode foundation
0.0	0.0	0.0	0.0
0.125	0.592276E-01	0.591543D-01	0.591253D-01
0.250	0.108474	0.108346	0.108292
0.375	0.140727	0.140568	0.140498

d) q = 6.75 lbs/in

Location x/L	Winkler	Modified Winkler	Mixed mode foundation
0.0	0.0	0.0	0.0
0.125	0.8884616E-01	0.893441D-01	0.892998D-01
0.250	0.162712	0.163656	0.163574
0.375	0.211090	0.212344	0.212237
0.500	0.227882	0.229248	0.229132

e) q = 9 lbs/in

Location x/L	Winkler	Modified Winkler	Mixed mode foundation
0.0	0.0	0.0	0.0
0.125	0.118455	0.119966	0.119906
0.250	0.216949	0.219677	0.219657
0.375	0.281454	0.285174	0.285028
0.500	0.303843	0.307884	0.307726

f) q = 15.75 lbs/in

Location x/L	Winkler	Modified Winkler	Mixed mode foundation
0.0	0.0	0.0	0.0
0.125	0.207297	0.214630	0.214517
0.250	0.379660	0.393300	0.393091
0.375	0.492545	0.510477	0.510204
0.500	0.531724	0.551182	0.550887

g) q = 22.5 lbs/in

Location x/L	Winkler	Modified Winkler	Mixed mode foundation
0.0	0.0	0.0	0.0
0.125	0.296138	0.313991	0.313816
0.250	0.542371	0.375550	0.575277
0.375	0.703635	0.747221	0.746800
0.500	0.759606	0.806886	0.806430

APPENDIX C

COMPUTER PROGRAM NOMENCLATURE

A complete listing and description of all variables used in the program is not practical. The items listed in this appendix are common to several areas of the program and will assist the reader in a study of the program. For convenience items are listed by variable type. The definition or function and dimension, if applicable, of each item is given.

A. INTEGER CONSTANTS

KA	maximum number of data of foundation modulus
KEI	maximum number of data of beam flexural rigidity
KQ	maximum number of loading conditions
KS	maximum number of various type of soils
KATEST	specified foundation modulus for test program only
KETEST	specified beam flexural rigidity for test program only
KQTEST	specified loading condition for test program only
ITMAX	maximum number of iterations
NBC	number of boundary conditions imposed
NCODE	code number, if NCODE is not equal to 1, the shear force in the foundation material is not taken into account

NDOF	number of degrees of freedom
NELEM	number of elements
NELMAX	maximum number of elements
NNOD	number of nodal points
NSIG	number of significant figures

B. REAL CONSTANTS

ALPHA	foundation modulus k
EI	beam flexural rigidity
Q	applied load
SSOIL	shear force in a soil foundation
TL	total length of the beam
X	beam element length

C. VECTORS

ASAV(10)	a set of foundation modulii
B(10)	coefficients of a least square fit polynomial
EF(10)	element force vector
EISAV(10)	a set of beam flexural rigidities
IDBC(20)	identification number of boundary conditions
PO(10)	a set of power p's in the term kV
QSAV(10)	a set of applied loads
SK(10)	a set of foundation modulii of natural soils
SS(10)	a set of shear forces of natural soils

SYSF(20)	system force vector
THETA(20)	slopes at the nodal points
V(20)	deflections at the nodal points
WA(600)	working vector

D. MATRICES

DC(4, 4)	C_{ik} - matrix, referred to Eqns. 23 and 24
DM(4, 4)	M_{ik} - matrix, referred to Eqns. 23 and 24
DN(4, 4, 4)	N_{ijk} - matrix referred to Eqns. 23 and 24
EK(4, 4)	element stiffness matrix K_{ik} referred to Eqns. 23 and 24
ICORR(20, 4)	correspondence table of local and global points
SYSC(20, 20, 20)	system N_{ijk} matrix, referred to Eqn. 24
SYSK(20, 20)	system C_{ik}, M_{ik} and K_{ik}- combined matrix, referred to Eqn. 24.

```
//TAM JOB (1714,0733,NS34),'      ',TIME=1
// EXEC FORTCLG,REGION=200K
//FORT.SYSIN DD *
C
C
C      THIS MAIN PROGRAM MAY BE USED FOR EITHER
C      LINEAR OR NONLINEAR PROBLEMS , ACCORDING TO
C      APPROPRIATE INSTRUCTION . FOR LINEAR
C      PROBLEMS,OMIT ALL 'IMPLICIT REAL*8(A-H,O-Z)'
C -    CARDS IN THE MAIN AS WELL AS IN ANY SUBROUTINE
C
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON EK(4,4),DC(4,4),DM(4,4),DN(4,4,4),
     1       ICORR(20,4),SYSK(20,20),SYSC(20,20,20),
     2       SYSF(20),EF(4),IDBC(20),B(10),
     3       NNOD,NELEM,NDOF,NBC,TL,AA1,AA2,
     4       X,EI,Q,ALPHA,SSOIL,A1,A2
      DIMENSION WA(600),THETA(20),V(20),
     1       PO(10),SK(10),SS(10),
     2       ASAV(10),EISAV(10),QSAV(10)
      EXTERNAL   AUX
    1 FORMAT('1')
C
C      THE NCODE NUMBER IS USED TO CONTROL THE
C      OUTPUT RESULTS DEPENDING ON THE TYPES
C      OF PROBLEMS . NCODE=1 FOR LINEAR
C      PROBLEMS . NCODE=2 FOR NONLINEAR PROBLEMS.
C
      NCODE=2
      CALL STORE(NDOF,NBC,TL,NELEM,
     1       KATEST,KETEST,KQTEST,KA,KEI,
     2       KQ,KS,EPS,NSIG,
     3       ASAV,QSAV,EISAV,PO,SS,SK,ICORR)
C
C      THE FOLLOWING PARAMETERS MAY BE USED IN
C      THE CASE OF UNIFORM BEAM WITH UNIFORM
C      LOAD AND MAY VARY ACCORDINGLY TO THE
C      BEAM FLEXURAL RIGIDITY , LOADING CONDITION
C      AND TYPES OF FOUNDATIONS.
C
      IP=1
      JP=1
      KP=1
      LP=1
      PRINT 1
      CALL INCHK(NELEM,NNOD,NBC,NDOF,
     1       X,TL,EI,Q,IP,JP,KP,ISOIL,
     2       SSOIL,ALPHA,NCODE,VSCALE,
     3       ASAV,EISAV,QSAV,SK,SS,IDBC,V,PO)
C
C      FOR THE CASE OF LINEAR PROBLEMS,REPLACE
C      'CALL STIFF2' CARD BY 'CALL STIFF1' CARD
C
      CALL STIFF2
      CALL LOAD
      CALL BOUND
C
C      FOR THE CASE OF LINEAR PROBLEMS,REPLACE
C      THE FOLLOWING CARDS BY
C      CALL SOLVE(SYSK,SYSF,V,NNOD)
C      CALL RESULT(NNOD,V,EI,IER,ITMAX,NCODE)
C
      NG=0
    4 CALL VGUESS(NNOD,NELEM,VSCALE,ITMAX,V)
      CALL ZSYSTM(AUX,EPS,NSIG,NNOD,V,ITMAX,WA,PAR,IER)
      IF(IER.EQ.0) GO TO 5
      NG = NG + 1
      IF(NG.LE.10) GO TO 4
    5 CALL RESULT(NNOD,V,EI,IER,ITMAX,NCODE)
      STOP
      END
```

```
      SUBROUTINE STORE(NDOF,NBC,TL,NELEM,
     1      KATEST,KETEST,KQTEST,KA,KEI,
     2      KQ,KS,EPS,NSIG,
     3      ASAV,QSAV,EISAV,PO,SS,SK,ICORR)
C
C     FOR LINEAR PROBLEMS,OMIT
C     'IMPLICIT REAL*8(A-H,O-Z)' CARD
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ASAV(10),QSAV(10),EISAV(10),SS(10),
     1      SK(10),PO(10),ICORR(20,4)
C
C     THE FOLLOWING PARAMETERS ARE USED FOR
C     TESTING THE CONVERGENCE OF THE PROGRAM.
C
      NELMAX=8
      KETEST=1
      KQTEST=1
      KATEST=1
      EPS=1.0D-06
      NSIG=5
C
C     READ IN ALL INFORMATION NEEDED TO THE
C     ENTIRE FIELD OF THE PROBLEM.
C
      READ(5,10) KA,KEI,KQ,NELEM
      READ(5,3) NDOF,NBC,KPO,KS,TL
    3 FORMAT(4I5,G15.3)
      READ(5,4) (PO(I),I=1,KPO)
      READ(5,4) (ASAV(I),I=1,KA)
      READ(5,4) (QSAV(I),I=1,KQ)
      READ(5,4) (EISAV(I),I=1,KEI)
      READ(5,4) (SK(I),I=1,KS)
      READ(5,4) (SS(I),I=1,KS)
    4 FORMAT(4G15.3)
C
C     READ IN THE CORRESPONDENCE  TABLE
C     BETWEEN LOCAL AND GLOBAL COORDINATES
C
      DO 11 I=1,NELMAX
   11 READ(5,10)ICORR(I,1),ICORR(I,2),ICORR(I,3),ICORR(I,4)
   10 FORMAT(4I5)
      RETURN
      END
```

```
      SUBROUTINE INCHK(NELEM,NNOD,NBC,NDOF,
     1     X,TL,EI,Q,IP,JP,KP,ISOIL,
     2     SSOIL,ALPHA,NCODE,VSCALF,
     3     ASAV,EISAV,QSAV,SK,SS,IDBC,V,PO)
C
C     FOR LINEAR PROBLEMS,OMIT
C     'IMPLICIT REAL*8(A-H,O-Z)' CARD
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION ASAV(10),EISAV(10),QSAV(10),
     1    IDBC(20),V(20),PO(10),SK(10),SS(10)
      VSCALE=1.D-06
      Q=QSAV(IP)
      EI=EISAV(JP)
      IF(NCODE.NE.1) GO TO 7
C
C     THESE TWO FOLLOWING STATEMENTS IS DEALING
C     WITH NATURAL SOILS
C
      SSOIL=SS(ISOIL)
      ALPHA=SK(ISOIL)
      GO TO 9
C
C     SSOIL  MAY HAVE A CERTAIN VALUE IF THE
C     SHEAR FORCE IN THE FOUNDATION MATERIAL
C     IS TAKEN INTO ACCOUNT.
C
    7 SSOIL=0.0
      ALPHA=ASAV(KP)
    9 NNOD=2*(NELEM+1)
      X=TL/(2*NELEM)
      PRINT 8,NNOD,NELEM,NBC,NDOF,X
    8 FORMAT(5X,' NUMBER OF NODAL POINTS=',I5,/,/,
     15X,'NUMBER OF ELEMENTS=',I5,/,/,
     25X,'NUMBER OF BOUNDARY CONDITIONS=',I5,/,/,
     35X,'NUMBER OF DEGREES OF FREEDOM=',I5,/,/,
     45X,'ELEMENT LENGTH=',1X,D15.6,////)
      BETA=(ALPHA/(4.*EI))**.25
C
C    THE FOLLOWING BOUNDARY CONDITIONS MAY NOT
C     BE NECCESARY IN GENERAL AND CAN BE
C     READJUSTED IN THE SUBROUTINE BOUND.
C
      IDBC(1)=1
      ICBC(2)=NNOD
      V(1)=0.0
      V(NNOD)=0.0
      RETURN
      END
```

```
      SUBROUTINE STIFF2
      IMPLICIT REAL*8(A-H,O-Z)
C
C    THIS SUBROUTINE IS USED TO COMPUTE
C    THE ELEMENT STIFFNESS MATRICES AND THE
C     SYSTEM STIFFNESS MATRIX FOR NONLINEAR
C     PROBLEM , ACCORDING TO THE FORMULA (   )
C     ON PAGE .......
C
      COMMON EK(4,4),DC(4,4),DM(4,4),DN(4,4,4),
     1       ICORR(20,4),SYSK(20,20),SYSC(20,20,20),
     2       SYSF(20),EF(4),IDBC(20),B(10),
     3       NNOD,NELEM,NDOF,NBC,TL,AA1,AA2,
     4       X,EI,Q,ALPHA,SSOIL,A1,A2
      DO 5 I=1,NNOD
      DO 5 J=1,NNOD
      SYSK(I,J)=0.0D+0
      DO 5 K=1,NNOD
    5 SYSC(I,J,K)=0.0D+0
      DO 6 N=1,NELEM
C
C     THE FOLLOWING FUNCTIONS  SFUNCT , EFUNCT ,
C     B2 , B3 ARE DEALING WITH VARIABLE SHEAR
C     FORCE IN THE FOUNDATION MATERIALS,VARIABLE
C     BEAM FLEXURAL RIGIDITY AND NONLINEAR
C     REACTION OF THE FOUNDATION . THE REACTION
C     OF THE FOUNDATION MAY BE WRITTEN AS
C     R(X) = B2*V + B3*V**2
C
      SSOIL=SFUNCT(N,X)
      EI=EFUNCT(N,X)
      B(1)=0.0D+0
      B(2)=B2(N,X)
      B(3)=B3(N,X)
      SSOIL=SSOIL/EI
      A1=ALPHA/EI
      AA1=A1*B(2)
C
C     THE FOLLOWING IS THE BEAM ELEMENT
C     STIFFNESS MATRIX
C
      EK(1,1)=12.D+00*(1.D+00/X**3)
      EK(1,2)=6.D+00*(1.D+00/X**2)
      EK(1,3)=-12.D+00*(1.D+00/X**3)
      EK(1,4)=6.D+00*(1.D+00/X**2)
      EK(2,2)=4.D+00*(1.D+00/X)
      EK(2,3)=-6.D+00*(1.D+00/X**2)
      EK(2,4)=2.D+00*(1.D+00/X)
      EK(3,3)=12.D+00*(1.D+00/X**3)
      EK(3,4)=-6.D+00*(1.D+00/X**2)
      EK(4,4)=4.D+00*(1.D+00/X)
      EK(2,1)=EK(1,2)
      EK(3,1)=EK(1,3)
      EK(3,2)= EK(2,3)
      EK(4,1)=EK(1,4)
      EK(4,2)=EK(2,4)
      EK(4,3)=EK(3,4)
```

```
C
C     THE FOLLOWING IS THE ELEMENT FOUNDATION
C     STIFFNESS MATRICES , ASSOCIATED TO THE
C-    LINEAR TERMS
C
      DC(1,1)=1.2D+00/X
      DC(1,2)=0.1D+00
      DC(1,3)=-DC(1,1)
      DC(1,4)=DC(1,2)
      DC(2,1)=DC(1,2)
      DC(2,2)=0.133333D+00*X
      DC(2,3)=-DC(1,2)
      DC(2,4)=-0.033333D+00*X
      DC(3,1)=DC(1,3)
      DC(3,2)=DC(2,3)
      DC(3,3)=DC(1,1)
      DC(3,4)=DC(2,3)
      DC(4,1)=DC(1,4)
      DC(4,2)=DC(2,4)
      DC(4,3)=DC(3,4)
      DC(4,4)=DC(2,2)
      DM(1,1)=.371429D+00*X
      DM(1,2)=.052381D+00*X**2
      DM(1,3)=.128571D+00*X
      DM(1,4)=-.030952D+00*X**2
      DM(2,2)=.009524D+00*X**3
      DM(2,3)=.030952D+00*X**2
      DM(2,4)=-.007143D+00*X**3
      DM(3,3)=.371429D+00*X
      DM(3,4)=-.052381D+00*X**2
      DM(4,4)=.009524D+00*X**3
      DM(2,1)=DM(1,2)
      DM(3,1)=DM(1,3)
      DM(4,1)=DM(1,4)
      DM(3,2)=DM(2,3)
      DM(4,2)=DM(2,4)
      DM(4,3)=DM(3,4)
C
C     THE CORRESPONDING COMPONENTS OF ALL
C     STIFFNESS MATRICES ASSOCIATED TO THE LINEAR
C     TERMS MAY BE ADDED AS FOLLOWS
C
      DO 1 I=1,NDOF
      DO 1 J=1,NDOF
      DC(I,J)=SSOIL*DC(I,J)
      DM(I,J)=AA1*DM(I,J)
    1 EK(I,J)=EK(I,J)+DC(I,J)+DM(I,J)
```

```
C
C     THE FOLLOWING ARE THE COMPONENTS OF THE
C     MATRIX ASSOCIATED TO THE NONLINEAR TERMS
C
      DN(1,1,1)=.307143D+00*X
      DN(2,1,1)=.038492D+00*X**2
      DN(3,1,1)=.064286D+00*X
      DN(4,1,1)=-.017061D+00*X**2
      DN(2,2,1)=.006349D+00*X**3
      DN(3,2,1)=.013889D+00*X**2
      DN(4,2,1)=-.003571D+00*X**3
      DN(3,3,1)=.064286D+00*X
      DN(4,3,1)=-.013889D+00*X**2
      DN(4,4,1)=.003175D+00*X**3
      DN(2,2,2)=.001191D+00*X**4
      DN(3,2,2)=.003175D+00*X**3
      DN(4,2,2)=-.000794D+00*X**4
      DN(3,3,2)=.017061D+00*X**2
      DN(4,3,2)=-.003571D+00*X**3
      DN(4,4,2)=.000794D+00*X**4
      DN(3,3,3)=.307143D+00*X
      DN(4,3,3)=-.038492D+00*X**2
      DN(4,4,3)=.006349D+00*X**3
      DN(4,4,4)=-.001191D+00*X**4
      DN(1,2,1)=DN(2,1,1)
      DN(1,3,1)=DN(3,1,1)
      DN(2,3,1)=DN(3,2,1)
      DN(1,4,1)=DN(4,1,1)
      DN(2,4,1)=DN(4,2,1)
      DN(3,4,1)=DN(4,3,1)
      DN(1,1,2)=DN(2,1,1)
      DN(1,2,2)=DN(2,2,1)
      DN(1,3,2)=DN(3,2,1)
      DN(1,4,2)=DN(4,2,1)
      DN(2,1,2)=DN(2,2,1)
      DN(2,3,2)=DN(3,2,2)
      DN(2,4,2)=DN(4,2,2)
      DN(3,1,2)=DN(3,2,1)
      DN(3,4,2)=DN(4,3,2)
      DN(4,1,2)=DN(4,2,1)
      DN(1,1,3)=DN(3,1,1)
      DN(1,2,3)=DN(3,2,1)
      DN(1,3,3)=DN(3,3,1)
      DN(1,4,3)=DN(4,3,1)
      DN(2,1,3)=DN(3,2,1)
      DN(2,2,3)=DN(3,2,2)
      DN(2,3,3)=DN(3,3,2)
      DN(2,4,3)=DN(4,3,2)
      DN(3,1,3)=DN(3,3,1)
      DN(3,2,3)=DN(3,3,2)
      DN(3,4,3)=DN(4,3,3)
      DN(4,1,3)=DN(4,3,1)
      DN(4,2,3)=DN(4,3,2)
      DN(1,1,4)=DN(4,1,1)
      DN(1,2,4)=DN(4,2,1)
      DN(1,3,4)=DN(4,3,1)
      DN(1,4,4)=DN(4,4,1)
      DN(2,1,4)=DN(4,2,1)
      DN(2,2,4)=DN(4,2,2)
      DN(2,3,4)=DN(4,3,2)
      DN(2,4,4)=DN(4,4,2)
      DN(3,1,4)=DN(4,3,1)
      DN(3,2,4)=DN(4,3,2)
      DN(3,3,4)=DN(4,3,3)
      DN(3,4,4)=DN(4,4,3)
      DN(4,1,4)=DN(4,4,1)
      DN(4,2,4)=DN(4,4,2)
      DN(4,3,4)=DN(4,4,3)
```

```
C
C     THE SYSTEM STIFFNESS MATRIX IS OBTAINED
C     BY COLLECTING THE ELEMENT STIFFNESS
C     MATRICES OVER ALL ELEMENTS OF THE SYSTEM
C
      AA2=A1*B(3)
      DO 4 K=1,NDOF
      DO 4 J=1,NDOF
      DO 4 I=1,NDOF
    4 DN(I,J,K)=AA2*DN(I,J,K)
      DO 7 I=1,NDOF
      NS1= ICORR(N,I)
      DO 7 J=1,NDOF
      NS2= ICORR(N,J)
      SYSK(NS1,NS2)=SYSK(NS1,NS2) + EK(I,J)
      DO 7 K=1,NDOF
      NS3= ICORR(N,K)
    7 SYSC(NS1,NS2,NS3)=SYSC(NS1,NS2,NS3) + DN(I,J,K)
    6 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE  LOAD
C
C     FOR THE LINEAR PROBLEMS , USING SINGLE
C     PRECISION,THE 'IMPLICIT REAL*8(A-H,O-Z)'
C     CARD MUST BE OMITTED
C
      IMPLICIT REAL*8(A-H,O-Z)
C
C     THIS SUBROUTINE IS USED TO COMPUTE
C     THE  ELEMENT  FORCE  VECTOR
C     AND THE SYSTEM FORCE VECTOR
C
      COMMON EK(4,4),DC(4,4),DM(4,4),DN(4,4,4),
     1       ICORR(20,4),SYSK(20,20),SYSC(20,20,20),
     2       SYSF(20),EF(4),IDBC(20),B(10),
     3       NNOD,NELEM,NDOF,NBC,TL,AA1,AA2,
     4       X,EI,Q,ALPHA,SSOIL,A1,A2
      DO 1 I=1,NNOD
    1 SYSF(I)=0.0D+00
      DO 3 N=1,NELEM
      EF(1)= .5D+00*X
      EF(2)= .083333D+00*X**2
      EF(3)= .5D+00*X
      EF(4)= -.083333D+00*X**2
      Q=QFUNCT(N,X)-A1*B(1)
      EF(1)=Q*EF(1)
      EF(2)=Q*EF(2)
      EF(3)=Q*EF(3)
      EF(4)=Q*EF(4)
      DO 2 NT=1,NDOF
    2 EF(NT)=EF(NT)/EFUNCT(N,X)
      I=ICORR(N,1)
      J=ICORR(N,2)
      K=ICORR(N,3)
      L=ICORR(N,4)
      SYSF(I)=SYSF(I)+ EF(1)
      SYSF(J)=SYSF(J) + EF(2)
      SYSF(K)=SYSF(K) + EF(3)
    3 SYSF(L)=SYSF(L) + EF(4)
      RETURN
      END
```

```
      SUBROUTINE BOUND
C
C     FOR LINEAR PROBLEMS,THE 'IMPLICIT REAL*8
C     (A-H,O-Z)' CARD MUST BE OMITTED
C
      IMPLICIT REAL*8(A-H,O-Z)
C
C     BEFORE SOLVING THE ALGEBRAIC EQUATION SYSTEM
C     THE BOUNDARY CONDITIONS MUST BE APPLIED
C
      COMMON EK(4,4),DC(4,4),DM(4,4),DN(4,4,4),
     1      ICORR(20,4),SYSK(20,20),SYSC(20,20,20),
     2      SYSF(20),EF(4),IDBC(20),B(10),
     3      NNOD,NELEM,NDOF,NBC,TL,AA1,AA2,
     4      X,EI,Q,ALPHA,SSOIL,A1,A2
      DO 3 N=1,NBC
C
C     FOR FREE-FREE ENDS BEAM IT IS DESIGNED
C     TO SKIP THE FIRST BOUNDARY CONDITION
C     WHICH IS RELATED TO THE DEFLECTION AT THE
C     END . THE SLOPE AT THE MIDDLE , WHICH
C     IS ZERO BY SYMMETRY,MUST BE IMPOSED.
C     FOR OTHER TYPES OF BOUNDARY
C     CONDITIONS,THIS SUBROUTINE IS NOT TRUE.
C
      IF(N.EQ.1) GO TO 3
      KK=IDBC(N)
      DO 4 K=1,NNOD
      SYSK(KK,K)=0.0D+00
      SYSK(K,KK)=0.0D+00
      DO 4 I=1,NNOD
      DO 4 J=1,NNOD
      SYSC(KK,I,J)=0.0D+00
    4 SYSC(KK,J,I)=0.0D+00
      SYSC(KK,KK,KK)=0.0D+00
      SYSF(KK)=0.0D+00
      SYSK(KK,KK)=1.D+00
    3 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE CURFIT(LP,PO,B)
C
C     USING LSQPL2 LIBRARY SUBROUTINE , THE
C     IRRATIONAL POWER CURVE WITH P NOT EQUAL TO
C     AN INTEGER IS REPLACED BY A LEAST SQUARE
C     BEST FIT SECOND ORDER POLYNOMIAL
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION B(10),WI(100),Y(100),DELY(100),
     1        PO(10),SB(100),F2(100),X(100),ER(100)
      REAL*8 TITLE(10)/10*'      '/
      DATA WI/100*1.D0/
      NMAX=-2
      P=PO(LP)
      IF(P.EQ.2.D+00) GO TO 10
      N=5
      DO 2 I=1,N
      FLOATI=I
      X(I)=FLOATI
    2 F2(I)=X(I)**P
      CALL LSQPL2(N,NMAX,X,F2,WI,Y,DELY,B,SB,TITLE)
      DO 6 I=1,N
      IF(F2(I).LE.1.D-06) GO-TO 7
      ER(I)=DELY(I)/F2(I)
      GO TO 6
    7 ER(I)=0.0D+00
    6 CONTINUE
      ERROR=ER(1)
      DO 8 I=2,N
      IF(DABS(ER(I)).LE.ERROR) GO TO 8
      ERROR=DABS(ER(I))
    8 CONTINUE
      PRINT 9,ERROR
    9 FORMAT(//,10X,'THE MAX RELATIVE ERROR IS=',D15.7)
      GO TO 11
   10 B(1)=0.0D+00
      B(2)=0.0D+00
      B(3)=1.D+00
   11 RETURN
      END
```

```
      SUBROUTINE FSOIL(B)
      IMPLICIT REAL*8(A-H,O-Z)
C
C     THE EXPERIMENTAL DATA ON A TYPICAL
C     NATURAL SOIL,I.E.,REACTION OF FOUNDATION
C     VERSUS LOADING CONDITION FROM TABLE......
C     AND FIGURE......,ARE APPRIMATED BY A LEAST
C     SQUARE BEST FIT SECOND ORDER POLYNOMIAL
C     USING LSQPL2 LIBRARY SUBROUTINE
C
      DIMENSION B(10),WI(10),Y(10),DELY(10),
     1          SB(10),F2(10),X(10),ER(10)
      REAL*8 TITLE(10)/10*'        '/
      DATA WI/10*1.D0/
      NMAX=-2
      N=6
      DO 2 I=1,N
    2 READ(5,3) X(I),F2(I)
    3 FORMAT(2G15.6)
      CALL LSQPL2(N,NMAX,X,F2,WI,Y,DELY,B,SB,TITLE)
C
C     THE SEA SEDIMENT FOUNDATION IS MODELED
C     FROM INLAND SOIL DATA BY A SCALE 1:3000.
C     OMIT THE THREE FOLLOWING CARDS IN CASE
C     IN CASE OF INLAND FOUNDATION
C
      B(1)=B(1)*8.3D+00/3000.D+00
      B(2)=B(2)*8.3D+00/3000.D+00
      B(3)=B(3)*8.3D+00/3000.D+00
      RETURN
      END
```

```
      SUBROUTINE VGUESS(NNOD,NELEM,VSCALE,ITMAX,V)
C
C     THIS SUBROUTINE PROVIDES AN INITIAL
C     ESTIMATE FOR NONLINEAR PROBLEMS
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION V(20)
      ITMAX=100
C
C     THE MAIN PROGRAM IS DESIGNED IN
C     SUCH A WAY THAT IF ONE INITIAL ESTIMATE
C     DOES NOT GIVE SOLUTION,THE SCALE WILL BE
C     CHANGED,AND THE ITERATION RESTART AGAIN.
C     HOWEVER,AFTER 10 CYCLES,IF NO SOLUTION IS
C     OBTAINED,THE READER MAY CHANGE THE INITIAL
C     SCALE BY HAND IN THE SUBROUTINE INCHK
C
      VSCALE=1.5D+00*VSCALE
      TAU=0.0D+00
      NT=NELEM+1
      DO 22 I=1,NT
      II=2*I
      III=2*I-1
C
C     FOR SIMPLICITY , A SECOND ORDER POLINOMIAL
C     IS USED FOR ESTIMATE SOLUTION . THE READER
C     MAY DESIGN OTHER ESTIMATE FUNCTION
C     AS HE WHISHES
C
      V(II)=VSCALE*2.D+00*(4.D+00-TAU)
      V(III)=VSCALE*((8.D+00*TAU-TAU**2)+0.01D+00)
   22 TAU=TAU+1.D+00
      V(1)=0.0D+00
      PRINT 23,(V(I),I=1,NNOD)
   23 FORMAT(////,15X,'INITIAL GUESS :',///,(15X,D15.6,/))
      RETURN
      END
```

```
      SUBROUTINE DIFSOL
C
C     USING FINITE DIFFERENCE METHOD TO SOLVE
C     NONLINEAR PROBLEMS,THIS SUBROUTINE IS
C     USED STRICTLY FOR THE CASE OF UNIFORM BEAMS
C     WITH UNIFORM LOADS AND THE REACTION OF
C     FOUNDATION IS PROPORTIONAL TO THE  SQUARE
C     OF ITS DEFLECTION
C
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON EK(4,4),DC(4,4),DM(4,4),DN(4,4,4),
     1      ICORR(20,4),SYSK(20,20),SYSC(20,20,20),
     2      SYSF(20),EF(4),IDBC(20),B(10),
     3      NNOD,NELEM,NDOF,NBC,TL,AA1,AA2,
     4      X,EI,Q,ALPHA,SSOIL,A1,A2
      DIMENSION U(10),WD(100)
C
C     TWO BEAM MODELS ARE CONSIDERED :
C     EXTERNAL DIFF4 CORRESPONDS TO 4-ELEMENT
C     MODEL,AND EXTERNAL DIFF8 TO 8-ELEMENT MODEL
C
      EXTERNAL DIFF4
      EXTERNAL DIFF8
    1 FORMAT('1')
      ND=NELEM
      DD=ND
      NSIG=5
      EPS=1.D-06
      NCOUNT=1
      USCALE=1000.D+00
    3 ITMAX=100
      NCOUNT=NCOUNT+1
      TAU=0.0D+00
      USCALE=1.5D+00*USCALE
      DO 4 I=1,ND
      TAU=TAU+2.D+00/DD
    4 U(I)=USCALE*(TAU-.25D+00*TAU**2)
      IF(ND.GT.2) GO TO 5
      CALL ZSYSTM(DIFF4,EPS,NSIG,ND,U,ITMAX,WD,PAR,IER)
      GO TO 8
    5 CALL ZSYSTM(DIFF8,EPS,NSIG,ND,U,ITMAX,WD,PAR,IER)
    8 IF(NCOUNT.GT.30) GO TO 6
      IF(IER.NE.0) GO TO 3
    6 PRINT 7,IER,ITMAX
    7 FORMAT(////,15X,'IER=',I5//15X,'ITMAX=',I5)
      DO 9 I=1,ND
    9 U(I)=U(I)/EI
      PRINT 10
   10 FORMAT(////,15X,'F.D.M. RESULT',////)
      PRINT 11,(U(I),I=1,ND)
   11 FORMAT(15X,G20.6/)
      RETURN
      END
```

```
      SUBROUTINE STIFF1
C
C     THIS SUBROUTINE IS USED TO COMPUTE THE ELEMENT
C     STIFFNESS MATRICES AND THE SYSTEM STIFFNESS
C     MATRIX FOR LINEAR PROBLEMS
C
      COMMON EK(4,4),DC(4,4),DM(4,4),DN(4,4,4),
     1      ICORR(20,4),SYSK(20,20),SYSC(20,20,20),
     2      SYSF(20),EF(4),IDBC(20),B(10),
     3      NNOD,NELEM,NDOF,NBC,TL,AA1,AA2,
     4      X,EI,Q,ALPHA,SSOIL,A1,A2
      ALPHA=ALPHA/EI
      SSOIL=SSOIL/EI
      EK(1,1)=12.*(1./X**3)
      EK(1,2)=6.*(1./X**2)
      EK(1,3)=-12.*(1./X**3)
      EK(1,4)=6.*(1./X**2)
      EK(2,2)=4.*(1./X)
      EK(2,3)=-6.*(1./X**2)
      EK(2,4)=2.*(1./X)
      EK(3,3)=12.*(1./X**3)
      EK(3,4)=-6.*(1./X**2)
      EK(4,4)=4.*(1./X)
      EK(2,1)=EK(1,2)
      EK(3,1)=EK(1,3)
      EK(3,2)= EK(2,3)
      EK(4,1)=EK(1,4)
      EK(4,2)=EK(2,4)
      EK(4,3)=EK(3,4)
C
C     THE FOLLOWING ARE THE ELEMENT STIFFNESS
C     MATRICES ASSOCIATED TO THE FOUNDATION
C
      DC(1,1)=1.2/X
      DC(1,2)=0.1
      DC(1,3)=-DC(1,1)
      DC(1,4)=DC(1,2)
      DC(2,1)=DC(1,2)
      DC(2,2)=0.133333*X
      DC(2,3)=-DC(1,2)
      DC(2,4)=-0.033333*X
      DC(3,1)=DC(1,3)
      DC(3,2)=DC(2,3)
      DC(3,3)=DC(1,1)
      DC(3,4)=DC(2,3)
      DC(4,1)=DC(1,4)
      DC(4,2)=DC(2,4)
      DC(4,3)=DC(3,4)
      DC(4,4)=DC(2,2)
      DM(1,1)=.371429*X
      DM(1,2)=.052381*X**2
      DM(1,3)=.128571*X
      DM(1,4)= -.030952*X**2
      DM(2,2)=.009524*X**3
      DM(2,3)= .030952*X**2
      DM(2,4)=-.007143*X**3
      DM(3,3)=.371429*X
      DM(3,4)= -.052381*X**2
      DM(4,4)= .009524*X**3
      DM(2,1)=DM(1,2)
      DM(3,1)= DM(1,3)
      DM(4,1)= DM(1,4)
      DM(3,2)= DM(2,3)
      DM(4,2)=DM(2,4)
      DM(4,3)= DM(3,4)
```

```
C
C     SINCE ALL ELEMENT STIFFNESS MATRICES ARE
C     ASSOCIATED TO THE LINEAR TERMS , THEY
C     MAY BE ADDED AS FOLLOWS
C
      DO 1  I= 1,NDOF
      DO 1  J= 1,NDOF
    1 EK(I,J)=EK(I,J)+ALPHA*DM(I,J)+SSOIL*DC(I,J)
      DO 2 I=1,NNOD
      DO 2 J =1,NNOD
    2 SYSK(I,J)= 0.0
C
C     THE SYSTEM STIFFNESS MATRIX IS OBTAINED  BY
C     COLLECTING THE STIFFNESS MATRICES OVER ALL
C     ELEMENTS  OF  THE  SYSTEM
C
      DO 3 N= 1,NELEM
      DO 4 I=1,NDOF
      NS1= ICORR(N,I)
      DO 4 J= 1,NDOF
      NS2= ICORR(N,J)
    4 SYSK(NS1,NS2)= SYSK(NS1,NS2) + EK(I,J)
    3 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SOLVE(SYSK,SYSF,V,NNOD)
C
C     BASED ON GAUSSIAN ELIMINATION TECHNIQUE
C     THIS EQUATION SOLVER IS USED TO SOLVE
C     THE LINEAR ALGEBRAIC SYSTEM ONLY
C
      DIMENSION SYSF(20),SYSK(20,20),
     1          IV(20),SYSFP(20),V(20)
      DO 1 K=1,NNOD
    1 IV(K)=K
C
C     GAUSSIAN ELIMINATION
C
      NM1=NNOD-1
      DO 1004 I=1,NM1
      IP1=I+1
      ATOP=ABS(SYSK(I,I))
      IPRIME=I
      DO 1100 IP=IP1,NNOD
      IF(ABS(SYSK(IP,I)).LE.ATOP) GO TO 1100
      ATOP=ABS(SYSK(IP,I))
      IPRIME=IP
 1100 CONTINUE
      IF(IPRIME.EQ.I) GO TO 1200
      ISAV=IV(IPRIME)
      IV(IPRIME)=IV(I)
      IV(I)=ISAV
      DO 1300 J=1,NNOD
      TEMP=SYSK(IPRIME,J)
      SYSK(IPRIME,J)=SYSK(I,J)
 1300 SYSK(I,J)=TEMP
 1200 DO 1005 K=IP1,NNOD
      FACTOR=SYSK(K,I)/SYSK(I,I)
      SYSK(K,I)=FACTOR
      DO 1006 J=IP1,NNOD
      SYSK(K,J)=SYSK(K,J)-FACTOR*SYSK(I,J)
 1006 CONTINUE
 1005 CONTINUE
 1004 CONTINUE
      DO 9 I=1,NNOD
      K=IV(I)
    9 SYSFP(I)=SYSF(K)
      DO 10 J=1,NM1
      JP1=J+1
      DO 11 K=JP1,NNOD
      SYSFP(K)=SYSFP(K)-SYSK(K,J)*SYSFP(J)
   11 CONTINUE
   10 CONTINUE
C
C     B A C K     S O L U T I O N
C
      SYSF(NNOD)=SYSFP(NNOD)/SYSK(NNOD,NNOD)
      DO 13 II=1,NM1
      I=NNOD-II
      IP1=I+1
      SUM=0.0
      DO 14 J=IP1,NNOD
   14 SUM=SUM+SYSK(I,J)*SYSF(J)
   13 V(I)=(SYSFP(I)-SUM)/SYSK(I,I)
      RETURN
      END
```

```
      SUBROUTINE RESULT(NNOD,SYSF,EI,IER,ITMAX,NCODE)
C
C     THIS SUBROUTINE IS USED TO PRINT THE
C     SOLUTION OF EITHER LINEAR OR NONLINEAR
C     PROBLEMS.FOR LINEAR PROBLEMS , OMIT
C     'IMPLICIT REAL*8(A-H,O-Z)' CARD
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION SYSF(20),V(20)
      IF(NCODE.EQ.1) GO TO 19
      PRINT 28 , IER,ITMAX
   28 FORMAT(//,15X,'IER=',1X,I2,////,
     115X,'ITMAX =',1X,I3,////)
   19 NN=NNOD/2
      DO 20 I=1,NN
      II=2*I
      JJ=2*I-1
   20 V(I)=SYSF(JJ)
      PRINT 30
   30 FORMAT(////,15X,'DEFLECTION :',//)
      PRINT 31,(V(I),I=1,NN)
   31 FORMAT( 5X,G20.6,/)
C
C     IF THE DEFLECTIONS AS WELL AS THE SLOPES
C     AT THE NODAL POINTS ARE DESIRED ,
C     ADD THE FOLLOWING CARD
C     PRINT 31,(SYSF(I),I=1,NNOD)
C
      RETURN
      END
```

```
C         THE FUNCTION AUX WHICH IS TRANSLATED
C         FROM EQUATION .....PAGE.....,GENERALLY
C         REMAINS   THE SAME.BUT THE READER HAS TO
C         REDESIGN OTHER EXTERNALS , SUCH AS QFUNCT,
C         EFUNCT,SFUNCT,B2,B3,WHICH MAY VARY ,
C         DEPENDING ON LOADING CONDITIONS,BEAM
C         CHARACTERISTICS AND TYPES OF FOUNDATIONS
C
          DOUBLE PRECISION FUNCTION AUX(V,K,PAR)
          IMPLICIT REAL*8(A-H,O-Z)
          COMMON EK(4,4),DC(4,4),DM(4,4),DN(4,4,4),
         1     ICORR(20,4),SYSK(20,20),SYSC(20,20,20),
         2     SYSF(20),EF(4),IDBC(20),B(10),
         3     NNOD,NELEM,NDOF,NBC,TL,AA1,AA2,
         4     X,EI,Q,ALPHA,SSOIL,A1,A2
          DIMENSION V(20)
          SKTEMP=0.0D+00
          SCTEMP=0.0D+00
          DO 1 I=1,NNOD
          SKTEMP=SKTEMP+SYSK(K,I)*V(I)
          DO 1 J=1,NNOD
        1 SCTEMP=SCTEMP+SYSC(K,I,J)*V(I)*V(J)
          AUX=SKTEMP+SCTEMP-SYSF(K)
          RETURN
          END

          DOUBLE PRECISION FUNCTION QFUNCT(N,X)
          IMPLICIT REAL*8(A-H,O-Z)
          FLOATN=N
          XI=(FLOATN-.5D+00)*X
          IF(N.NE.1) GO TO 3
          QFUNCT=0.0D+00
          GO TO 4
        3 QFUNCT=((3.5D+00/3.D+00)+(1.D+00/150.D+00)*
         1     XI)*1.D+04
        4 RETURN
          END

          DOUBLE PRECISION FUNCTION B2(N,X)
          IMPLICIT REAL*8(A-H,O-Z)
          FLOATN=N
          XI=(FLOATN-.5D+00)*X
          IF(N.NE.1) GO TO 3
          B2=2000.D+00+40.D+00*XI
          GO TO 4
        3 B2=(100.D+00/15.D+00)*XI + (11000.D+00/3.D+00)
        4 RETURN
          END
          DOUBLE PRECISION FUNCTION B3(N,X)
          IMPLICIT REAL*8(A-H,O-Z)
          FLOATN=N
          XI=(FLOATN-.5D+00)*X
          B3=.5D+00*XI+300.D+00
          RETURN
          END

          DOUBLE PRECISION FUNCTION EFUNCT(N,X)
          IMPLICIT REAL*8(A-H,O-Z)
          FLOATN=N
          XI=(FLOATN-.5D+00)*X
          IF(N.NE.1) GO TO 3
          EFUNCT=1.D+11+(1.D+11/50.D+00)*XI
          GO TO 4
        3 EFUNCT=2.D+11
        4 RETURN
          END

          DOUBLE PRECISION FUNCTION SFUNCT(N,X)
          IMPLICIT REAL*8(A-H,O-Z)
          SFUNCT=0.0D+00
          RETURN
          END
```

```
      DOUBLE PRECISION FUNCTION DIFF4(U,K,PAR)
      IMPLICIT REAL*8(A-H,O-Z)
      CCMMON EK(4,4),DC(4,4),DM(4,4),DN(4,4,4),
     1      ICORR(20,4),SYSK(20,20),SYSC(20,20,20),
     2      SYSF(20),EF(4),IDBC(20),B(10),
     3      NNOD,NELEM,NDOF,NBC,TL,AA1,AA2,
     4      X,EI,Q,ALPHA,SSOIL,A1,A2
      DIMENSION U(10)
      GO TO (5,10),K
    5 DIFF4=1.536D+00*U(1)+0.001D+00*A2*(TL**4)*U(1)**2-
     11.024D+00*U(2)-0.001D+00*Q*(TL**4)
      RETURN
   10 DIFF4=-2.048D+00*U(1)+1.536D+00*U(2)+
     10.001D+00*A2*(TL**4)*U(2)**2-0.001D+00*Q*(TL**4)
      RETURN
      END
```

```
      DOUBLE PRECISION FUNCTION DIFF8(U,K,PAR)
      IMPLICIT REAL*8(A-H,O-Z)
      CCMMCN EK(4,4),DC(4,4),DM(4,4),DN(4,4,4),
     1      ICORR(20,4),SYSK(20,20),SYSC(20,20,20),
     2      SYSF(20),EF(4),IDBC(20),B(10),
     3      NNOD,NELEM,NDOF,NBC,TL,AA1,AA2,
     4      X,EI,Q,ALPHA,SSOIL,A1,A2
      DIMENSION U(10)
      GO TO (5,10,15,20),K
    5 DIFF8=5.D+00*U(1)-4.D+00*U(2) + U(3) +
     1(X**4)*(A1*B(1)+AA1*U(1)+AA2*U(1)**2)
     2     -Q*X**4
      RETURN
   10 DIFF8=-4.D+00*U(1)+6.D+00*U(2)-4.D+00*U(3)+
     1U(4)+(X**4)*(A1*B(1)+AA1*U(2)+AA2*U(2)**2)
     2     -Q*X**4
      RETURN
   15 DIFF8= U(1)-4.D+00*U(2)+7.D+00*U(3)-4.D+00*U(4)+
     1(X**4)*(A1*B(1)+AA1*U(3)+AA2*U(3)**2)
     2     -Q*X**4
      RETURN
   20 DIFF8= 2.D+00*U(2)-8.D+00*U(3)+6.D+00*U(4)+
     1(X**4)*(A1*B(1)+AA1*U(4)+AA2*U(4)**2)
     2     -Q*X**4
      RETURN
      END
```

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Beams on nonlinear foundations.